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ABSTRACT

The third volume of this proceedings contains full research articles. Papers include: (1) "A longitudinal study of children's fraction representations and problem-solving behavior" (G.A. Goldin and C.B. Passantino); (2) "Psychology students' conceptions of a statistics course" (S. Gordon, J. Nicholas, and K. Crawford); (3) "Choosing a visual strategy: The influence of gender on the solution process of rotation problems" (N. Gorgorio); (4) "Discourse in an inquiry math elementary classroom and the collaborative construction of an elegant algebraic expression" (B. Graves and V. Zack); (5) "Number processing: Qualitative differences in thinking and the role of imagery" (E. Gray and D. Pitta); (6) "Identification of Van Hiele levels of reasoning in three-dimensional geometry" (G. Guillen); (7) "Working with 'the discipline of noticing': An authenticating experience" (T. Hardy and D. Wilson); (8) "Classifying processes of proving" (G. Harel and L. Sowder); (9) "Seeing, doing and expressing: An evaluation of task sequences for supporting algebraic thinking" (L. Healy and C. Hoyles); (10) "The role of prior conceptions in teachers' responses to staff development: A synopsis of case studies of three middle school mathematics teachers" (T.A. Herrera); (11) "The use of levels of subordination to help students gain fluency in mathematics" (D. Hewitt); (12) "An analysis of the development of pupil understanding in group work activities using multimedia" (B. Hudson); (13) "Some issues in assessing proceptual understanding" (M. Hunter and J. Monaghan); (14) "The development of language about function: An application of Van Hiele levels" (M. Isoda); (15) "Hitomi's meaning construction of table and algebraic expression of proportion during instruction: A case study" (K. Ito-Hino); (16) "Communicating teacher's metaknowledge through lessons" (H. Iwasaki); (17) "A study of teacher enquiry into the processes of mathematics teaching" (B. Jaworski); (18) "Using children's probabilistic thinking to inform instruction" (G.A. Jones, C.A. Thornton, C.W. Langrall and T.A. Mogill); (19) "Coming to know about 'dependency' within a dynamic geometry environment" (K. Jones); (20) "Somali children learning mathematics in Britain: A conflict of cultures" (L. Jones); (21) "From measurement to conjecture and proof in geometry problems--students' use of measurements in the computer environment" (K. Kakihana, K. Shimizu, and N. Nohda); (22) "Mathematics teachers' training: Some remarks about the role of

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self-identification" (M. Kaldrimidou and M. Tzekaki); (23) "To have or not to have mathematical ability, and what is the question" (R. Karsenty and S. Vinner); (24) "Invisible angles and visible parallels which bring deconstruction to geometry" (E. Kopelman); (25) "Research on the complementarity of intuition and logical thinking in the process of understanding mathematics: An examination of the two-axes process model by analyzing an elementary school mathematics class" (M. Koyama); (26) "Application of reification theory in translating verbal expressions and statements into algebraic expressions" (B. Kutscher); (27) "Measures of teachers' attitudes towards mathematical modeling" (J.I. Kyeleve and J. Williams); (28) "Innovation-in-practice: Teacher strategies and beliefs constructed with computer-based exploratory classroom mathematics" (C. Kynigos); (29) "The implementation of curriculum policy on classroom organization in primary mathematics in Cyprus" (L. Kyriakides); (30) "Partitioning and unitizing" (S.J. Lamon); (31) "Simultaneously assessing intended, implemented and attained conceptions about the gradient" (A.C. Leal, A.B. Ciani, I.G. Do Prado, L.F. Da Silva, P.R. Linardi, R.R. Baldino and T.C.B. Cabral); (32) "When change becomes the name of the game: Mathematics teachers in transition to a new learning environment" (I. Levenberg and A. Sfard); (33) "The competition between numbers and structure" (L. Linchevski and D. Livneh); (34) "Situated intuitions, concrete manipulations and the construction of mathematical concepts: The case of integers" (L. Linchevski and J. Williams); (35) "Secondary pupils' translations of algebraic relationships into everyday language: A Hong Kong study" (F. Lopez-Real); (36) "Letting go: An approach to geometric problem solving" (E. Love); (37) "Learning to formulate equations for problems" (M. MacGregor and K. Stacey); (38) "Origins of students' interpretations of algebraic notation" (M. MacGregor and K. Stacey); (39) "Mathematical beliefs behind school performances" (M.L. Malmivuori and E. Pehkonen); (40) "Preservice secondary mathematics teachers' beliefs: Two case studies of emerging and evolving perceptions" (J.A. Malone); (41) "On the notion of function" (J. Mamona-Downs); (42) "Reasoning geometrically through the drawing activity" (M.A. Mariotti); (43) "Thinking about geometrical shapes in a computer based environment" (C. Markopoulos and D. Potari); (44) "The quest for meaning in students' mathematical modelling activity" (J.F. Matos and S. Carreira); (45) "The role of imagery and discourse in supporting the development of mathematical meaning" (K. McClain and P. Cobb); (46) "The origins and developments of the NCTM professional standards for teaching mathematics" (D. McLeod); (47) "Mathematics and the sign" (O. McNamara); (48) "Student's early algebraic activity: Sense making and the production of meanings in mathematics" (L. Meira); (49) "Performance and understanding: A closer look at comparison word problems" (I. Mekhmandarov, R. Meron, and I. Peled); (50) "Graphing calculators and pre-calculus: An exploration of some aspects of students' understanding" (V.M. Mesa and P. Gomez); (51) "On the utilization of encoding procedures on the treatment of geometrical problems" (A.L. Mesquita); (52) "Children's developing multiplication and division strategies" (M. Mitchelmore and J. Mulligan); and (53) "Children's concepts of turning: Dynamic or static?" (M. Mitchelmore and P. White). (ASK)

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PME 20

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Edited by

Luis Puig
Angel Gutiérrez

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Dept. de Didàctica de la Matemàtica

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RESEARCH REPORTS

(continuation)

A LONGITUDINAL STUDY OF CHILDREN'S FRACTION REPRESENTATIONS AND PROBLEM-SOLVING BEHAVIOR[1]

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As part of a longitudinal study of children's mathematical development we analyzed videotapes of 20 elementary-school children solving problems in two carefully-structured task-based interviews administered one and one-half years apart. Here we describe and discuss three individual students' behaviors, with attention to the external representations and models they employed or constructed in attempting non-standard problems in the domain of fractions. From our observations we seek to draw preliminary inferences about the development of these children's understandings of fractions.

As mathematics education research focuses more on children's processes of constructing meaning, researchers have sought to describe in greater detail how particular mathematical concepts develop. Task-based interviews are being used increasingly to explore students' developing mathematical understandings (Davis, 1984). The observed problem-solving behaviors of children permit conjectures or theories about the internal representations and conceptual understandings giving rise to those behaviors (Lesh, Post & Behr, 1987; Goldin, 1987, 1988, 1992).

The research reported here is part of a descriptive longitudinal study conducted at Rutgers University on the development of mathematical understandings in children in grades 3-6 (Goldin *et al.*, 1993). Five highly structured task-based interview scripts were created to investigate how children's internal systems of mathematical representation develop over time, and the role of such representations in their changing conceptual knowledge and problem-solving capabilities. Two interviews in the sequence, #2 and #5, were designed to focus on fraction representations. We shall describe some behaviors of three individual children. We make use of the external representations and models they employ or construct as they attempt to solve non-standard problems in the domain of fractions to draw preliminary inferences about their developing understandings of fractions.

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[1] The research reported in this paper was partially supported by a grant from the U.S. National Science Foundation (NSF), "A Three-Year Longitudinal Study of Children's Development of Mathematical Knowledge," directed by Robert B. Davis and Carolyn A. Maher. The opinions and conclusions expressed are those of the authors, and do not necessarily reflect the views of the NSF.

Research questions

The overall research questions we investigate fall into the following four areas: (1) *External representations and models*: What particular fraction representations or models are in evidence? Which of these seem to endure over time? Does the individual child evidence use of a set model, a linear model, some sort of region or area model, a division model, or a model of a different sort? What external modes of representation—words, notational symbols, pictures, enactive behavior, gestures, recall of daily life experiences, etc.—predominate, and how do these change over time? (2) *Strategies and problem-solving heuristics*: In solving problems about fractions, what problem-solving strategies, heuristics, or metacognitive activity can we infer or conjecture? How do the children's internal, strategic representations facilitate or impede (a) problem solution and (b) conceptual understanding of fractions? (3) *Making connections*: How stable are the students' constructs? To what extent or under what circumstances do children change or abandon representations or models, or make new connections among representations or models? What evidence can we find of students making translations or transformations among fraction representations, of semiotic acts assigning meanings in one representation to configurations in another? In particular, do their models or representations change or interact in ways that suggest "reconceptualization cycles" or "local conceptual development" (Lesh, Hole, & Post, to be published) during the problem-solving episodes? (4) *Learning and teaching*: What are the links between "model-eliciting activities" (Lesh & Kaput, 1988) and the processes of instruction and assessment? What can be learned or conjectured from exploratory observations of children's problem solving in the domain of fractions about how to foster overall development of mathematical competency, and deeper understandings of fractions?

Design and administration of the interviews

Scripts for task-based clinical interviews were designed and developed by a team including the authors and other graduate students working toward advanced degrees at Rutgers University, under the leadership of the first author. All members of the team had professional teaching experience in mathematics or elementary education.

Each script follows explicit principles in its construction (Goldin, 1993). The child is asked a series of questions of increasing mathematical difficulty, so that the final task is one that can be attempted by all the children, but is challenging even to the most skillful. During the interview, the child engages in free problem-solving with minimal input from the clinician (except for prompts such as, "Can you tell me more about that?" asking for explanations of what the child is doing or descriptions of what the child is thinking). All student efforts are "accepted" without preconceived notions about appropriate solution strategies, and (with a few, specified exceptions) without distinguishing between "wrong" or "right" answers. The clinician typically asks follow-up questions to responses without indicating their correctness. When an impasse is reached, the clinician offers structured heuristic suggestions, in accordance with each script, and again allows for free, uninterrupted problem solving by the child. The suggestions continue until the child solves the problem, or (after an interval of time) the clinician moves on to another section of the interview. Each interview is designed to take approximately 45 minutes (one class period) to administer. Materials are available for student use, depending on the problems posed in the interview: paper and pencil, markers, chips or other manipulatives, paper cut-outs, string, rulers, calculators, etc. The abundance of flexibly-applicable materials allows the researcher to observe external representations made or used by the students, and to explore connections among representations (Lesh, Post, & Behr, 1987). Each interview includes retrospective questions and questions to explore the child's affect during problem solving.

Each script was revised several times by the development team. Revisions were guided by mock interviews with each other, followed by interviews with individual children of the developers' personal acquaintance, and finally a videotaped, pilot clinical study with children in a nearby urban elementary school. The pilot sessions permitted critical evaluation of the draft scripts, and training for clinicians through mutual critiques of interviewing techniques.

Of 22 children in the longitudinal study, 20 participated in both of the interviews that focused on fraction concepts, #2 and #5. Two videotapes were made at each of these interviews. One camera focused on the interaction between the child and the clinician, showing their faces; the second camera focused on the student's work. Interview #2 was conducted in January and February 1993, when the children were in the middle of fourth or fifth grade (ages 9-11 years); interview #5 in the spring of 1994, when the same children were at the end of fifth or sixth grade. The students came from a cross-section of New Jersey communities: from one school in a predominantly blue-collar, "working class" community, one school in a suburban, upper middle-class district, and two urban schools. Though the group included girls and boys of differing backgrounds, ethnicity, and ability levels, it was not drawn as a stratified random sample. The study should be regarded as a set of exploratory or investigative case studies, not as an experiment yielding valid generalizations for a wider population.

We next describe briefly just those portions of the two task-based interview scripts for which children's behaviors are discussed here. The full interview scripts are available from the authors on request.

Task-Based Interview #2: Early in this interview the child is asked several questions related to his or her understanding of one-half and one-third: ° When you think of one-half, what comes to mind? ° When you think of one-third, what comes to mind? The purpose here is to invite freely described representations of fractions, without yet suggesting a specific context. Other parts of interview #2 provide opportunities for the child to describe a region and/or a set model for fractions, e.g.: ° Suppose you had twelve apples. How would you take one-half/one third? Several different cut-out shapes are presented, and for each the child is asked: ° Here is a shape. How would you take one-half/one-third? ° Why is this one-half/one-third? ° Are there any other ways to take one-half/one-third? The student's ability to write a fraction and understanding of notational meaning are then explored: ° Can you write the fraction one-half/one-third? ° What does this fraction mean to you? Another activity focuses on the way students work with an array of objects. The overall goal in this part of interview #2 is to investigate various fraction representations and models the child spontaneously uses or describes, and to observe the child's facility in making connections or moving from one representation to another. Each main question is followed by dialogue designed to elicit more specific descriptions or concrete models with the provided materials. Observation and analysis of interview #2 also becomes baseline data related to interview #5.

Task-Based Interview #5: This interview begins by asking the child an open-ended question similar to (but more general than) those asked in interview #2: ° When you think of a fraction, what comes to mind? The described representations are later to be compared to those elicited at the beginning of interview #2, when the child was asked about one-half and one-third. Next the student is engaged in a discussion about fractions, and the kinds of things he or she has done with fractions in and out of school. A paper is then shown with five fractions written on it, all with numerals in large bold print, in vertical format with a horizontal fraction bar: one-half, one-third, two-thirds, three-fourths, and four-sixths. Questions asked include: ° What fractions do you see here? ° Can you explain to me what the fractions mean? ° Why are they written this way? ° Could you show me what they mean using some of the materials? ° Which fraction is the smallest (largest) fraction in the group? ° Are there any fractions in this group that are the same size? Two additional sheets of paper are shown successively to the student, one with pictorial representations that can be interpreted as corresponding to various fractions, and another with "improper" fractions written numerically. The child is subsequently asked to show one-third and then one-fourth of a cut-out circle. A series of questions explores the child's understandings, including connections made among such external representations and the stability of the child's described fraction constructs. Later in this interview a series of problems are posed, each with the possibility of some fractional interpretation. In one of these, an unmarked piece of wood, measuring 1"x1"x5", is placed in front of the student (recall that a variety of other materials, including a ruler, a length of string, a calculator, pencils and markers, etc. remain on the table): ° Pretend this is a stick of butter. You need a tablespoon of butter to make a cake. You don't have a measuring spoon, but you know that there are eight tablespoons in a stick of butter. Here is the butter. How could you find exactly one tablespoon?

Analysis and comparison of the two task-based interviews for all 20 children with respect to the four categories of research questions above is presently under way. Here we summarize some preliminary observations for three of the students.

Preliminary observations and inferences

Fernando: In fifth grade, Fernando (age 10) mentions "two pieces" as essential to one-half, and "three pieces" as essential to one-third. In finding one-half of a shape, he only asserts that the pieces have to be equal when discussing the circle. He also indicates that pieces have to be the same shape. In discussing thirds, he mentions that the pieces have to be the same size for the square ("you cut them out to see if they are the same size"), but denies that this is important when discussing vertical slices into three parts that he has made of the circle and the flower cut-outs. What matters for Fernando with the latter shapes seems to be only the number of pieces (3). When shown a wedge shape aligned with the circumference of the circle cut-out (having 1/3 the circle's area), and asked if this could represent one-half or one-third, Fernando responds affirmatively: it could represent one-half, because there are two pieces; it could represent one-third, because there *would be* three pieces if you drew the other line. When shown another wedge shape aligned with the circumference of the circle cut-out (having 1/6 the circle's area), he agrees it represents one-sixth, but says it is too small to be one-half or one-third: it could not be one-third, because "you need more than three pieces to complete the circle". When the same wedges are placed inside the circle, rather on the circumference, Fernando made some interesting adjustments in his thinking. A wedge with half the area could now represent one-third, because the bottom section of the circle looks like one-third (as he drew it when he "sliced" the circle into three pieces); the wedge with 1/3 the area could still represent one-third if you put it back on the edge of the circle; and the wedge with 1/6 the area could still represent one-sixth, or maybe one-fifth, because when it gets put back at the edge, five or six of them could fit into the circle. At this point in his development Fernando recognizes the number of pieces as critical to the fraction; he seems to think that the pieces should be the same shape, though this may not always be true; he is uncertain about whether they need to be the same size. He switches easily to a set model when asked to take one-half or one-third of twelve apples (he does this by dividing), or one-half or one-third of an array of twelve shapes (where he disregards their colors and shapes).

In sixth grade (age 12), when asked to show one-third, he uses the same model, again slicing the circle and shading the right-most "third". It still does not trouble him that the pieces are different sizes or shapes. A marked change, though, is that he now recognizes that the pieces can be split to form more slices; so that 2/6 would be the same as 1/3, or 2/8 would be the same as 1/4. This is clearly related to his demonstrated ability to find equivalent fractions, by multiplying or dividing the numerators and denominators by the same number, using 2/2. or 3/3 or 4/4, etc. He volunteers that you could show one-third by taking one out of three circles. He solves the butter problem by measuring 12 1/2 cm, then dividing the 12 by 8 to get 1 1/2 cm. He measures off one tablespoon only, and says the others would be the same size because he figured it out. It is not clear if he recognizes this as an approximate solution. But in the butter problem Fernando does recognize that "same size" can be important; after you get eight pieces, you can "cut them all and measure them on top of each other".

Graham: In fourth grade (age 10), Graham demonstrates flexibility in representing fractions meaningfully. What comes to mind with one-half is half the population of Rhode Island; for one-third he thinks of a pie with three pieces in it "because that's what we usually say - one slice of a pie". He is versatile in showing halves and thirds of shapes, emphasizing that the pieces must be the same size, even if they are not the same shape. He delights in making squiggly shapes, and knows that there are "infinity ways" to divide the circle or the flower in half by making diameters. He changes flexibly to a set model when finding one-half or one-third of twelve apples or twelve mixed shapes.

In fifth grade (age 11), Graham never mentions irregular shapes. When asked what comes to mind when you think of a fraction, he says "just the fraction, two numbers with a line in the middle." He easily represents and orders the fractions, and uses the algorithm for determining equivalence. He has also learned to divide and to form decimal numbers. This seems to interfere with his solving the butter problem, because he divides 5 by 8 and ends up with a number in the "hundredths and thousandths", which he cannot use. He estimates "six tenths", but then has problems finding this amount in inches. He estimates where this is on the ruler. The clinician says, "At about the 5/8 mark?" and Graham agrees. Then he goes on to try centimeters, about 1 3/4 cm, but this is too large so he gives up. He says they don't do problems like this in school because "when the teacher gives it to you it's usually easy to solve." It is difficult to escape the implication that schooling is diminishing rather than enhancing Graham's flexibility of representation, even as his algorithmic proficiency increases.

Jack: When asked in fifth grade (age 10) what comes to mind when you think of one-half, Jack says he thinks of half of a circle, because it's the easiest thing to cut in half; for one-third he thinks of a rectangle, because it's easiest to split into thirds. When finding halves of shapes he emphasizes going through the middle to make two equal parts. He makes three vertical slices to find thirds of a square, but says it is impossible to find thirds of the circle or the flower because when you slice them the slices are not equal. Curiously, however, he immediately recognizes the wedge-shaped third of a circle and says that it could be used to represent one-third. He volunteers that three of the wedges whose size is 1/6 of the circle would make one-half, and that two of them would make one third. The wedges needed to be in position at the edge of the circle for Jack to recognize this: "They don't represent anything unless you move them back to the edge." He easily finds one-half or one-third of the apples or the shapes, but insists that each third or half of the shapes have the same number of circles and flowers, as "a circle does not equal a flower".

In sixth grade (age 12), when Jack thinks of fractions, he mainly thinks of the numbers. He compares fractions mostly by the algorithmic procedure of finding a number that "goes into both". When asked to explain he just laughs, and says "It works and it's right". When pressed further, however, Jack does make rectangular regions which he compares visually. In fifth grade he could only imagine making slices of the circle, but in sixth grade he immediately makes an upside-down Y to trisect it. He says he can't think of any other way to do it. It is difficult to infer internal representations from Jack's behavior in solving the butter problem, as he offers little verbally during the twenty minutes he works on it. He appears to be desperately trying to find a number that works. When pressed he says that one inch is too big, one-half inch is too small, three-quarters is too big. After some time the clinician says, "So it's bigger than one-half and smaller than three-quarters". Then Jack realizes that 1/2 = 4/8 and 3/4 = 6/8, so maybe it's 5/8! He appears thrilled with this discovery. It is interesting that Jack is "ready" to figure out 5/8 in the context of his heuristic problem solving, while Graham (above) does not react when "given" the answer 5/8, even though this puts into words the location he is indicating on the ruler.

Conclusion

Our observations suggest that in some situations, increased technical capability of symbolic mathematical representation of fractions does not imply increased flexibility of application or depth of conceptual understanding. There is evidence that for these children exploring various concrete and imagistic representations of fractions in greater depth would enhance their conceptual development.

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PSYCHOLOGY STUDENTS' CONCEPTIONS OF A STATISTICS COURSE

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ABSTRACT

We report in this paper the preliminary results from a study to investigate the conceptions of a compulsory statistics course held by university psychology students. Phenomenographic research methods were used to analyse responses to questionnaires administered to 270 students. A set of five categories of description for the students' conceptions of their statistics course were identified. Relationships were found between the students' conceptions, their attainments in assessment tasks and their willingness to study statistics. The results suggest that a majority of students view statistics as essentially disconnected from other knowledge. Moreover, a narrowly algorithmic approach was reinforced by assessment requirements.

What do psychology students think they are learning when they are required to study statistics at university? What conceptions of statistics do their attainments in statistics examinations reflect? Is there a relationship between students' willingness to study statistics and their conceptions? What connections do they see between statistical knowledge and their broader concerns? These are some of the questions we attempt to answer in this paper for a group of 270 students who were studying statistics as a compulsory component of second year Psychology. The paper is based on ongoing research into students' orientations into learning statistics at university (Gordon, 1993; 1995; In Press).

The prominence of statistics in university courses has generated considerable research into statistical education in the last fifteen years. This research shows that many students have difficulties with and misconceptions about statistical ideas (See, for example, Garfield and Ahlgren, 1988; Green, 1994). Many studies have focussed on reforms in statistics education by suggesting new and improved ways of teaching statistics (Eg Garfield, 1993; Hawkins et al, 1992; Romero et al, 1995).

In contrast, our perspective focuses on what is learned rather than what is taught. In order to take the standpoint of the students, that is, take a "second order perspective", the research adopts a phenomenographic approach (Marton, 1986; Crawford, Gordon, Nicholas & Prosser, 1994). This approach is described by Marton (1988) as "a research specialisation to study the different understandings or conceptions of phenomena in the world around us." Such an approach views phenomena systemically and avoids the boundaries between person and context. It is consistent with a Vygotskian view that there is no assumption of a duality between self and contest; between thinking and acting (Vygotsky, 1978). The use of a phenomenographic approach to this research has allowed us to describe the experience of learning statistics at university from the point of view of the students themselves. We have attempted to give our students a voice, a voice not normally heard in statistical education.

METHOD OF RESEARCH

Over 270 second year Psychology students, at the University of Sydney, completed a questionnaire on their conceptions of the statistics component of the course and their attitudes and approaches to learning statistics. The survey was completed approximately halfway through semester 1, that is, in week 12 of a 21 week semester. The questionnaire included the following open ended question, designed to elicit students' own conceptions of the statistics they were currently studying.

What in your opinion is this statistics course about? Please explain as fully as possible.

The first stage in the analysis of the data was to identify a set of qualitatively different categories of description to the open ended question. This involved the following procedure:

1. An initial set of categories was identified, by two independent researchers reading and classifying the entire set of responses to the above question.
2. The two researchers then compared and discussed the categories and agreed on a draft set of categories.
3. They, together with a third researcher, independently classified 30 of the responses in terms of this set of categories.
4. The individual classifications of the three researchers were compared and a final set of clear statements of each category was agreed upon.
5. All 270 responses were then classified accordingly.
6. All responses were discussed and agreement reached on any classifications that did not match.

The students were asked whether they would have studied statistics if it had not been compulsory to do so. Their responses to the questionnaire were then analysed to explore the relationships between students' conceptions of the statistics course and their attainment in tests and examinations during the first semester, their expressed willingness to study the statistics and gender.

RESULTS

Categories of Description of Students' Conceptions of the Statistics Course

The phenomenographic analysis of the responses to the question yielded a set of five qualitatively different categories. A label for each of the categories is given below. Labels are followed by descriptions of that category and illustrative excerpts from students' written responses.

1) NO MEANING

Students' responses that indicated perceptions of the course as meaningless or unconnected to their goals in learning psychology, worthless or set by the university as a means to confuse or "cull" less able students, were classified in this category.

For example a student responded to the question as follows: *Trying to confuse me*

2) PROCESSES or ALGORITHMS

Responses were classified in this category if:

a) The student's responses to the question consisted of a list of one or more statistical procedures such as *hypothesis testing* or *tabulating data.*

b) The student's perception of the course was reported in terms of an input-output machine or black box. That is, the response indicated a perception of the course as being about mechanical processes or coding.

Examples: *Number crunching*

and *Statistical results from experiments.*

... It's not necessary, considering computers do all the work.

3) MASTERY OF STATISTICAL CONCEPTS AND METHODS

Responses were classified in the third category if students reported their perception of the course in terms of competence or proficiency in the methods of statistics. Typical responses mentioned some or all of the following class exercises: analysing or interpreting given data, coming to conclusions on the basis of decontextualised information, solving practice exercises. In short, reading and/or understanding statistical information in isolation from the rest of their studies of psychology.

For example a student wrote: *To give us the basics in statistics.*

Another wrote: *Determining the results of experiments in the correct manner ...*

4) MASTERY AND A TOOL FOR GETTING RESULTS IN REAL LIFE

Responses in this category included notions of proficiency in statistical methods but also referred to the use of statistics in conducting research or its use in society.

An example of a response indicating the perception of the course as providing a tool was: *using statistics to apply it to experiments we will use later on in careers in psychology. A practical course.*

5) The final category was labelled A WAY OF CRITICAL THINKING.

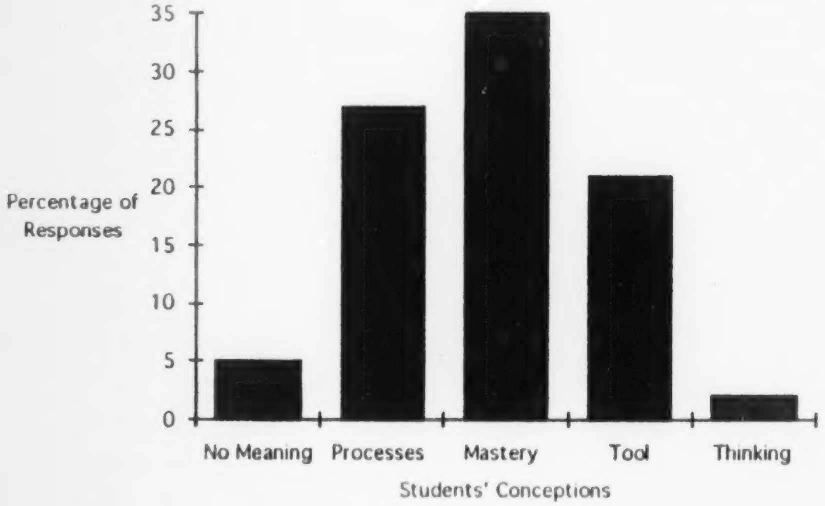
Responses were classified in this category if they included the idea of statistics as a tool and, in addition, referred to the statistics course as being about a (mathematical, scientific) way of critically evaluating findings, or organising, communicating and assessing findings.

An example of a response in this category was: *Understanding how numbers can provide evidence for or against some hypothesis you are testing. As a way of ensuring the validity & reliability of your own research methods. To understand how numbers can be used to falsify data/conclusions.*

Distribution of Responses

Figure 1 below indicates the distribution of responses into the five categories.

FIGURE 1: Distribution of Students' Conceptions of the Statistics Course



Interestingly, a large number (11%) of the students omitted to answer the question or responded in a way that indicated a reluctance to think about it. It appears that many students find it difficult or are unwilling to articulate their conceptions.

Relationship to Other Variables

Performance in Tests and Examinations

We first consider how students' reported conceptions of the statistics course relate to their performances in assessments in the first semester. Students had three assessment tasks. There were two class tests which were open book exercises, in which students were asked to show all working, and a multiple choice examination. No books or notes could be used in the examination. Relationships to the examination marks and the average of the two class tests (which will be referred to as the class mark) are reported separately.

The means for the students' performances are shown in Table 1 and Figure 2.

TABLE 1: Average Assessment Marks for Each Conception Group

Conception	Means for Class Marks	Means for Examination Marks
No Meaning	64% (N=9)	47% (N=9)
Processes	59% (N=55)*	57%(N=54)
Mastery	59% (N=73)	49% (N=69)
Tool	58% (N=43)	56% (N=41)
Thinking	80% (N=3)	53% (N=4)

*N differs in some cases, as some students did not write both the examination and the class tests.

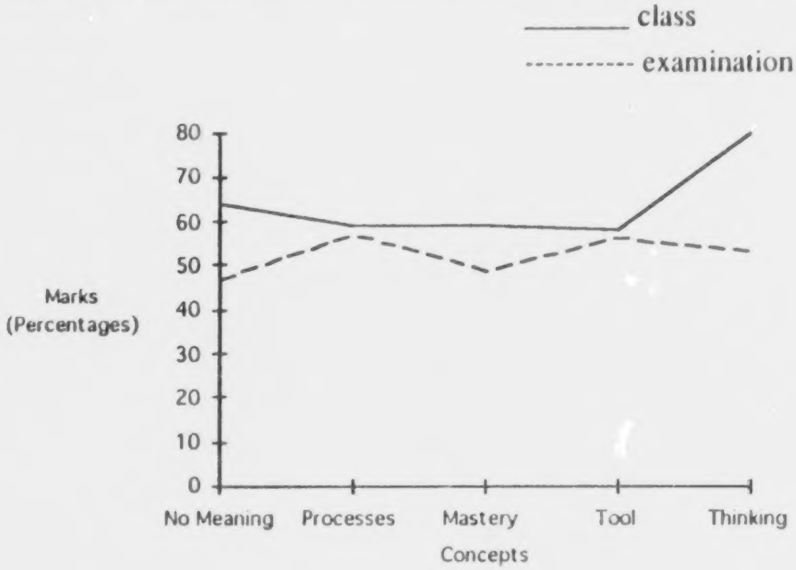
Figure 2 below shows that students performed better in the class tests than in the examination. The correlation between these students' class marks and examination results is 0.6.

When comparing the attainments of the different concept groups on the two different forms of assessment (problem solving tests and multiple choice examination), an interesting pattern emerges. No statistically significant differences were found between students' average attainments on their class tests for the three largest Conception groups: *Processes, Mastery* and *A Tool.* With the exception of the three students whose conceptions of the statistics course related to a way of thinking, no mean increase in marks was gained by students who conceived of the course in other than algorithmic terms. Thus, increased effort to make meaning were not rewarding under the conditions of the assessment of the class exercises.

The group of students who reported their conceptions of the statistics course as algorithmic *Processes* obtained higher marks in the multi choice examination that any other group. Indeed the *Processes* group perfomed significantly better on the semester 1 examination than those who reported their conceptions in terms of

Mastery (paired t =8.11, p=0.0001). This result implies that their conception was associated with an expedient approach to learning, in terms of course grades. For the vast majority of students, a purely algorithmic or mechanical conception of statistics was reinforced by successful assessment results.

FIGURE 2: Mean Marks for Class Tests and Examination for Conception Groups



Willingness to Study Statistics

Seventy four percent of the students surveyed answered "No" to the question: *Would you study statistics if it were not a requirement of your psychology course?*

The table below shows the differences in the perceptions of the course between those that responded in the affirmative and those who expressed an unwillingness to study statistics. Almost 80% of the "Yes" students reported their conceptions in terms of mastering the methods and concepts of statistics and/or using it as a tool. Few (14%) of these students reported thinking of the course as being about statistical procedures. However, over a third of the "No" students reported their conceptions in terms of algorithms and processes. Not surprisingly, none of the students who expressed the opinion that the statistics course had no meaning for them expressed a willingness to study statistics.

TABLE 2: Students' Concepts Versus Their Willingness to Study Statistics

Percentage (Number) of Students

	No Meaning	Processes	Mastery	Tool	Thinking
"No" (N=181)	7% (13)	36% (65)	36% (66)	20% (36)	1% (1)
"Yes" (N=65)	0	14% (9)	45% (29)	34% (22)	8% (5)

The increased interest in statistics of the minority group who felt positive about studying statistics was reflected in their grades. On average, the "Yes" group achieved higher marks on both forms of assessment.

Gender

For both sexes, *Mastery* was the modal category. However, Table 3 below shows that a considerably higher proportion of females than males reported thinking about the course as providing a tool which could be used in the future. On the other hand, a larger proportion of males than females evidently perceived the statistics course to be about mechanical processes or statistical procedures. In general, males performed better on the multiple choice examination.

TABLE 3: Students' Concepts Versus Their Sex

Percentage (Number) of Students

	No Meaning	Processes	Mastery	Tool	Thinking
Females (N=201)	3% (7)	25% (50)	36% (72)	25% (51)	2% (4)
Males (N=75)	8% (6)	29% (22)	33% (25)	8% (6)	3% (2)

DISCUSSION

There has been some concern about the outcomes of statistics education. This research has focussed on student conceptions of a compulsory statistics course — their point of view. The majority of the psychology students who were surveyed were unwillingly studying statistics at university. Most reported learning mechanical procedures or decontextualised statistical concepts and methods. Further, these conceptions appear to be reinforced by success in formal assessment tasks. The minority group of students who expressed a greater willingness to participate in statistics courses reported more thoughtful and personally meaningful conceptions of statistics. Their increased motivation was reflected in higher marks. However, for most students, a lack of understanding and interest was no deterrent to their successful completion of class exercises nor reflected in grades.

Although the course was a component of wider studies in psychology, less than a quarter of the students expressed an awareness of connections between statistical knowledge and applications in psychology or any other field. If we regard statistics as a useful and human endeavour, university educators will need support to ensure that students receive meaningful experiences of doing statistics that go beyond mere "*number crunching*" so that they cannot imagine that "*computers do all the work*". Such support will require more time and better resources than are usually allocated.

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Choosing a Visual Strategy: the influence of gender on the solution process of rotation problems.

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Abstract

Visualisation in geometry requires the mental manipulation of visual imagery. Most previous studies related to visualisation have focused on the processes involved in solving mathematical problems in general. Little research has been published that takes into account the occurrence of visual processing when children work on spatial tasks, presented through figural stimuli. This report discusses the relationship between achievement and strategies used by students aged 12-16 when tackling transformation problems involving spatial rotations. It compares, as sample groups and not as individuals, the mathematical behaviour of boys and girls and not solely their performance. The results suggest that gender is not enough an explanatory variable when analyzing the solving processes involved in spatial tasks, at least when students face tasks whose geometric focus is a rotation.

Spatial abilities, visual processing and gender.

One of the most studied aspects among those related to mathematical abilities is the analysis of differences among individuals. However, there are two different ways of viewing a person's various mathematical abilities. One way is to consider the level of accomplishment in some given tasks, which have some common characteristics determined in advance. The second way, is to consider the individual's cognitive traits that facilitate the solving processes of those tasks.

The construct visualisation appears not only in most of the studies dealing with spatial abilities, but also in many researches related to the solving processes of mathematical problems in general. On the studies concerned with spatial abilities, visualisation, even if not always having the same meaning, is often related to the idea of achievement. The origins of the research considering visualisation as a trait of the solving processes of mathematical problems in general, was the individual's characterization, presented in Krutekskii's work (1976), from their mathematical cast of mind considered as something different from their level of spatial ability. Bishop (1983), taking as a starting point the idea that it is impossible to establish a single definition of spatial ability, and trying to focus attention on the significant learning processes, suggests we consider two different abilities (op. cit., p. 181): the ability of interpreting figural information (IFI), and the ability of visual processing (VP). Bishop emphasizes those aspects related to processes over those related to the stimuli form and refers to visual processing in the mathematical context, in its broadest sense, and therefore in a context where visual stimuli are not always needed.

Since the beginning of research related to spatial tasks, many studies have attempted to analyze the variables that influence the degree of achievement that students demonstrate in those kinds of tasks. Many and varied aspects have been studied: gender, cultural influence, curriculum content, material manipulation, and so on. Through the years, the amount of research that demonstrated a superior achievement for boys than girls in spatial tasks is so impressive that to refer to them all would be cumbersome. The evidence seems so enormous that it is difficult not to conclude that a characteristic of spatial tasks is masculine superiority. Nevertheless, there is research pointing out that more variables need to be taken into account: age of sample people (Hall and Hoff (1988), Nash (1979), ethnic origins (Van Leeuwen (1978), cultural backgrounds (Hanna 1989), or what is understood as spatial tests (Eliot and Smith (1982), Halpern (1989), Shuard (1982) and Wattanawaha (1977)).

Most of the studies related to gender differences in spatial abilities are concerned with achievement, and regretfully not with the differences in solving strategies or processes. The amount of research dealing with differences related to gender in the solving processes of spatial tasks remains very small. Some of these studies conclude that even if there are not gender differences in performance, it should not be assumed that strategies used by individuals of both genders are the same (Newcombe et al. (1989), Tartre (1990)). The analysis of others do not lead us to identify any characteristic of the mental procedures of boys and girls that allows us to suppose the existence of any differences between strategies and solving processes of both genders (Battista (1990), Lohman (1979), Presmeg (1985)).

Some of the research quoted above refers to the solving processes of mathematical tasks in general. However, the content of the visualisation of abstract relationships has a different nature from that of the visual processing of geometric facts. Actually, most of the studies that analyze cognitive processes related to visualisation, are interested in the solving processes of mathematical problems in general. Little has been published regarding the analysis of the solving processes of spatial tasks, presented through figural stimuli, and taking into account the possibility of using or not using visual processing.

The solving processes of rotation tasks.

This study deliberately proposes to use the construct *spatial processing ability* instead of the construct visual processing ability, in order to clearly state the difference between the ability to solve any situation by means of a visual processing strategy, and the ability to cope with a spatial task, having already visual roots, using any kind of strategies.

In the present research, *spatial processing ability* is understood *as the ability needed to fulfill the combined mental operations required to solve a spatial task. It includes not only the ability to imagine spatial objects, relationships and transformations, but also the ability to encode them into verbal or mixed terms. It also includes the ability not only to manipulate the visual images of spatial facts, but also the ability to solve the tasks using processes that are not merely visual.*

Obviously, the *spatial processing ability* so defined, even if described with a singular term, has plural meanings. Spatial processing ability includes at least as many different abilities as many spatial transformations one may imagine. The present study focussed on one of its aspects, rotation.

The research (Gorgorió 1995), I am referring to, analyzed and characterized the strategies used by a sample of students, aged 12 16, when dealing with geometric tasks that required a spatial rotation. On this report, I present the results concerning the comparison of the mathematical behaviour of boys and girls, as sample groups and not as individuals, from the point of view of strategies used during the solving processes and not solely their performance.

Taking as a starting point Burden and Coulson's study (1981), and modifying it to fit the present research goals, students' strategies were analyzed from three different standpoints: the origins and the organizing of the information used, the mental representation mode, and the focus of attention. Therefore, for every subject and for every task, one may speak of *structuring strategy*, *processing strategy* and *approaching strategy*, being not three different kind of cognitive strategies, but three different aspects of the student's solving strategy. Following, there is a short description of each category, for further details see Gorgorió (1996).

For the study of *structuring strategies*, the student's cognitive strategy was considered from the standpoint of the different mental ways of facing the task, the mental organization, and source of the information used to cope with the task.

When analyzing *processing strategies*, the student's cognitive strategy was considered from the standpoint of its form of mental representation. The premise was taken that all mathematical problems imply reasoning or logic in their solving processes. Furthermore, all the tasks presented in the present research to sample students had a figural support on its presentation. Therefore, the fact that determined which kind of processing strategy the student used was the use or not the student made of visual images during the solving process, a fact that could only be elicited from students explanations and observation.

The analysis of the students' cognitive strategy considering its attention focus over the geometric object led to determine his or her *approaching strategy*.

Method.

Qualitative data obtained through clinical interview was used in the analysis of students' solving processes. Quantitative analysis was used also, in order to achieve the other goals of the study. Qualitative and quantitative analysis being complementary generated the results of research and contributed to the study's validity.

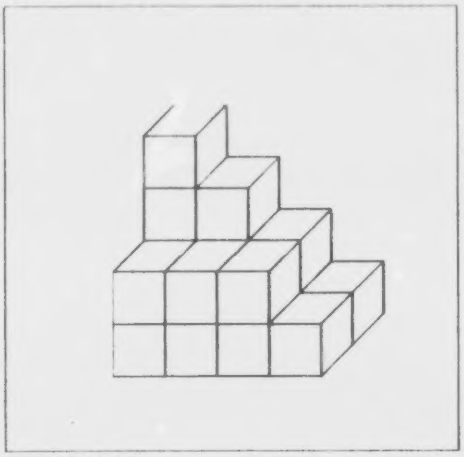
Nine tasks were presented to a sample of students to be solved during the interviews. The geometric demand of all the tasks was a spatial rotation. All the tasks were presented with visual support, using both real objects and 2-D representations of 3-D objects.

As one of the assumptions was that task characteristics influence students strategies, the tasks' statements were prepared carefully. Among the characteristics that were considered as being liable to modify or influence students strategies, the most significant turned out to be the *required action*. *Required action* is the action to be done by the subject in order to solve the task, in the sense established by Leinhardt et al. (Leinhardt et alt. (1990)). Among the tasks, there were 4 whose required action was of interpretation, that is to say, where the students had to react in front of a geometric action presented as accomplished, or to gain meaning from an object or a representation, without representing or drawing anything.

Three of the tasks of interpretation *(1 I, 2 I, 3 I)*, had the form of a multiple choice question, where students had to decide which was the correct answer by identifying objects being or not the same through rotation. Those tasks belonged to a test that has been created and validated in a previous study. The test content included some 3-D geometry items related to curricular content, and some other items to test the performance of students in spatial tasks in general.

The tasks whose required action was of construction were 5. In those tasks, students had either to draw or to construct, with wooden cubes, an object fulfilling the geometric request. Given the initial object, the student had to generate the final one, that means he or she had to apply a rotation, mentally or through manipulation, over it to generate a new one, real not imagined. Task 2 A, presented next, is an example of tasks of construction.

Construct, with the wooden cubes,
the object presented in the figure, as it would remain
after rotating it 180° over its base.



Task 2 A.

The tasks were administered to a sample of 21 students, aged 12 to 16, selected from a broadest sample of 645, from different types of schools, which had been administered the test previously mentioned. When selecting the sample to be interviewed, students' characteristics were diversified, taking into account theoretical conditions: gender, age and performance at the spatial test.

For every task, the interviews were prepared beforehand, planning a detailed sketch from the results and observations of a pilot experiment. Interviews were tape recorded, and drawings and objects made by the students were put away. Students' processes of drawing and construction were recorded through codified notes. During the interviews, the researcher also noted actions, movements and gestures made by the students that were considered to be hints of strategies used. Students were asked for the description of their solving processes once the task was accomplished. The transcription of all interviews, drawings and objects produced by students during the interview, and researcher's notes were the initial data.

Systemic networks, Bliss et al.(1983), were used to unfold, structure and reduce the data. Comparing the data corresponding to all the tasks, structured through networks, allowed the characterization and description of the different kind of strategies. Other goals required a quantitative analysis to be achieved. In such cases, for each task and for each category of strategies, tables were built summarizing the data. From the tables, the existence of some tendencies was observed. Further statistical analysis was used to decide which tendencies were enough significant to be considered. Broader results, for instance those concerning the differences among strategies used by boys and girls, were attained comparing quantitatively and qualitatively the evidence obtained through parallel processes done for each task and for each kind of strategy.

Results.

For each task, students' cognitive strategies were characterized as being structuring, processing and approaching strategies. Only a short characterization is presented here. In Gorgorió (1996), the reader will find a detailed description and some examples of the different types of strategies within each category.

The *structuring strategies* observed implied the student getting involved in the context of the situation, using information obtained from previous experience, or simplifying the task's structure.

Processing strategies were characterized as being visual or verbal. A processing strategy was considered to be ***visual*** when, from the student's explanations, it was clear enough that he or she had imagined some of the following aspects: the task's context, a rotation or a position's change of either the subject or the object. When saying that it was clear enough that a student had imagined any situation, it is meant that either the student had explicitly said he or she had imagined it, or it could be elicited from the student's explanations and observation. That would be the case of some students who said to be performing an action, a physical action, when they actually did not. A student's processing strategy was considered to be ***verbal*** when the student solved the task without imagining any situation, but relying on facts related to properties of 180° rotation, symmetry, congruence or using information belonging to the context.

Approaching strategies were characterized as being global or partial. An approaching strategy was considered to be ***global*** when the subject focussed his attention over the object or the situation considered as a whole: by comparing it with a real life object or situation, or by referring to the objects' congruence. It was considered to be *partial* when he or she focussed his or her attention on some parts of the object, taking into account some of the following aspects: the existence of significant parts, their characteristics, their relative position, or the elements resulting of splitting up the object.

In terms of gender differences, **some qualitative differences were observed among structuring and processing strategies used by boys and girls**, and **no difference appeared among approaching strategies** used by individuals of both genders. When such differences appeared, related to structuring or processing strategies, they depended on the required action of the task.

Concerning structuring strategies, when there were qualitative differences and the required action of the task was of interpretation, girls tended to use structuring strategies consisting of simplifying the task's structure, while boys did not use any structuring strategy. For instance, in one of the tasks of interpretation, where students had to compare four options among them, girls tended to take one of the options as a model, and compare the others with that one, while boys tended to deal with all four options simultaneously.

When the required action of the task was of construction, girls did not use any structuring strategy, and boys were distributed among those who relied on previous knowledge and those who did not use any approaching strategy. Boys relying on previous knowledge made use of

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information obtained from previous experiences or information which could explain the situation or helped to solve the task. For instance, in task 2 A presented before, one of the students (**A.B.**) said, without being asked and just before initiating the solving of the task, that he should take into account what changes take place when turning 180° an object over its base.

> **A.B.**: *I have to built up the object, say... the part behind has to be in front, and the right has to go to the left ... when building it up.*

In a similar way, differences appearing on the processing strategies used by boys and girls, depended on the required action of the task. Moreover, qualitative differences among processing strategies appeared only when the required action of the task was of interpretation. The following table presents, for each task, its type of required action, the existence of differences among genders concerning processing strategies used, and the significance of those differences.

Task	required action	difference boys/girls	significance
Streets	interpretation	NO	
1 I	interpretation	YES	90%
2 I	interpretation	NO	
3 I	interpretation	YES	90%
1 A	construction	NO	
2 A	construction	NO	
3 A	construction	NO	
1 B	construction	NO	
3 B	construction	NO	

From the table one observes that qualitative differences among processing strategies appeared only on two tasks, where the required action of the task was of interpretation: boys tended to use visual processing strategies and girls to use verbal processing strategies.

Concerning difficulties and errors, some differences between boys and girls had been observed. Those differences on the errors observed during the interviews corresponded to the ones observed on the large sampling test. Girls had more difficulties and made more errors when interpreting tasks' statements so for the verbal language referred to spatial facts and objects as for the representational code used. Girls made also more geometric errors than boys. A significant difference between the number of geometric errors of boys and girls had been found for three tasks. Girls tended to mistake a 180° rotation for a symmetry.

The differences between erros of both genders can be interpreted through boys and girls using different strategies. In the task of interpretation, where appeared significant differences among errors, those were related to the processing strategies used. Verbal processing strategies — which were on the most used by girls — led to a biggest number of errors due to missinterpretation of the task's statement. In the tasks of construction, differences are tied to a distinct use of structuring strategies. Structuring strategies on which the subject relies on previous experiences tended to lead to correct answers. The number of boys who used this kind of strategy is substantially bigger than that of girls. This fact could explain the differences favoring males, of the number of correct answers.

Conclusions.

The methodology used on a research very much conditions the nature of the results one arrives at. By large sample testing one may conclude general assertions, which, on the other hand, give only information about achievement and not processes. Qualitative analysis of data, obtained through interviewing a reduced sample, leads to obtain results of a descriptive nature about students' solving processes that can only be used to explain students behaviour on analogous situations. However, both methodologies may be used on a complementary way as in the present research, where part of the results obtained through statistical analysis of a big sample test, may be explained through the answers of a reduced sample.

Another important issue of the present research relates to the key role tasks' characteristics have as influencing factors of students' solving processes. The most relevant concerning gender differences is, probably, the required action. The studies considering spatial orientation ability as the achievement level on a spatial test, were presenting to students tasks of interpretation. In such kind of tasks, qualitative differences in the use boys and girls do of processing strategies have been found, and that different use led to differences in the answers' correctness. The researches studying solving processes, dealt with tasks of construction. In tasks of construction no differences have been observed neither in processing strategies, nor in approaching strategies of boys and girls. However, some differences on the structuring strategies used by boys and girls have been observed. Furthermore, when there are differences among boys' and girls' answers, those can be explained through that different use of structuring strategies.

Moreover, the results of this study add evidence to the fact that sex is not enough a differentiating variable when analyzing the solving processes of spatial tasks, for differences between genders are less than differences within genders, at least when students face tasks whose geometric demand is a rotation. It is one of the writer's beliefs that education should help students to overcome their difficulties, but should not force them to renounce their individual traits. Therefore I conclude quoting Clements and Battista (1992, p. 458) 'we should eventually be able to move beyond studying gender differences to the study of different cognitive profiles that underlie successful performance in geometry'.

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DISCOURSE IN AN INQUIRY MATH ELEMENTARY CLASSROOM AND THE COLLABORATIVE CONSTRUCTION OF AN ELEGANT ALGEBRAIC EXPRESSION

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This paper investigates how the discourse practices of two Grade 5 students mediated their reasoning processes in an inquiry mathematics classroom. The focus is on the collaborative exchange as a mechanism for conceptual change as the students engaged in a difficult problem-solving activity. Of particular interest is how students drew on their shared knowledge and interest to maintain the discussion and how the role of genuine inquiry within the talk resulted in the construction of what we have designated an elegant algebraic expression.

This paper investigates the discourse of inquiry as we examine how the discourse practices of two Grade 5 students mediated their reasoning processes in an inquiry mathematics classroom. Currently, the study of discourse holds an unprecedented high profile in research which investigates human cognition as an interaction of individual, social and cultural processes (Cole, 1991). Many researchers have focused broadly on the social and functional uses of language in society (cf., Halliday, 1975; John-Steiner, Smith & Panofsky, 1994; Vygotsky, 1978), while others have focused specifically on classroom discourse including the mathematics and science classroom (Ball, 1991; Green & Dixon, 1993; Lemke, 1991). In a recent article Hicks (1995) reviews aspects of discourse as an inherently social construct which mediates children's academic learning, and discusses the educational reforms in relation to discursive activities (e.g.,. NCTM, 1989). In regard to the 'appropriation' of mathematical discourse, children become schooled in the practice of mathematics as they learn to make connections between their own inventions and the conventions of the culture (Lampert, 1990; Cobb, Wood, & Yackel, 1993). At the same time as children learn about cultural tools such as algebraic generalization, they come to appreciate the power and authority inherent in those tools (Wertsch & Rupert, 1993).

In our investigation of communication in collaborative problem-solving exchanges, the work of Teasley and Roschelle (in press) is especially pertinent. Teasley and Roschelle define collaboration as "a coordinated, synchronous activity that is the result of a continued attempt to construct and maintain a shared conception of a problem." They identify this shared conceptual structure as the joint-problem-space which has two important features: 1) the joint-problem-space is constructed and maintained by means of conversation in the context of problem-solving activity; and 2) the joint-problem-space is the structure which enables the conversation about problem-solving to take place. The underlying assumption is that while overlap in meaning in the collaborators' common conception of the problem may be neither complete not certain, it is sufficient to lead to a gradual accumulation of shared concepts.

In this paper the episodes selected for study illustrate the students' search for meaning, and their appreciation of an elegant solution in terms of coherence, economy, and explanatory power. Their engagement with the ideas in their mathematics assignment seemed to us a prototype of what inquiry is. We will deal with their individual conceptualizations of the problem, as evidenced through their talk as we consider the question: "What does it mean for a cognitive process to occur both *in* and *between* individuals?" (Cole, 1991, p. 398-399). We concur with Vygotsky's view that the sign (word, diagram, algebraic notation) both represents a person's thinking, and transforms it (Wertsch & Toma, 1995, p. 163) and we adopt the premise that communication, activity and representation are mutually constitutive (Teasley & Roschelle, in press).

The school community and classroom setting

St. George's is a private, non-denominational school, with a middle class population of mixed ethnic, religious, and linguistic backgrounds; the population is predominantly English-speaking. The total class size in the 1993-1994 year was 25; the work, however, is always done in half-groups (12 or 13 children in each group) of heterogeneous ability. Problem-solving is at the core of the mathematics curriculum in this classroom.

The school and classroom learning site is a community of practice which Richards (1991) has called *inquiry math*; it is one in which the children are expected to publicly express their thinking, and engage in mathematical practice characterized by conjecture, argument, and justification (Cobb, et al., 1993, p. 98). The students have been tackling non-routine problems in diverse areas of the curriculum since their entry to the school, hence six years for many. For a number of the children, it is likely that academic discourse would be heard at home, as well as in school.

Mathematics class periods are 45 minutes, and twice a week are extended to 90 minutes. In addition to the in-class problem-solving sessions, each week the children also work on one challenging problem at home. They are expected to record their work and reflect on their strategies in a Math Log which serves as the initial basis of their group discussions in class. In class much of the session is conducted by the children as they discuss the problem first with a partner, then in a group of four consisting of two pairs, and finally with the entire group of twelve students. In this way each problem is examined on four separate occasions in multiple contexts.

The data

The children are videotaped throughout the school year on a rotating basis as they work in their groups. In addition to the videotape records, data sources include focused observations, student artifacts (math logs), teacher-composed questions eliciting opinions (written responses), and retrospective interviews.

The mathematical context of the problem/discussion

The focus in this paper is on the final problem, *Tunnels revisited*, in a series of 4 inter-related problems which are increasingly demanding. Below is the sequence of problems:

- #1 *Tunnels:* **'Nine prairie dogs need to connect all their burrows to one another in order to be sure that they can evade their enemy, the ferret. How many *tunnels* do they need to build?'** (February 7, 1994)
- #2 *Decagon Diagonals:* **How many diagonal lines can be drawn inside a figure with 10 sides?** (April 25, 1994)
- #3 *25-Sided-, 52-Sided Polygons*: **How many diagonals would there be in a 25-sided polygon? in a 52-sided polygon?** (May 16, 1994)
- #4 *Tunnels revisited:* **Can you write a number sentence or general rule for the *Tunnels* problem?** (May 24, 1994)

In an earlier paper Zack (1995) described how the children in her 1993-1994 grade 5 class worked together to arrive at an understanding of generalization. The two joint authors of this paper, Vicki Zack, a teacher-researcher in her homeroom classroom for the past 7 years, and Barbara Graves, a university researcher, have extended the investigation by examining how two of those children applied that knowledge to solve an additional challenging problem. The two boys, Jeff and Micky, both managed to generate a general rule to solve problem #3, "How many diagonals would there be in a 25-sided polygon? in a 52-sided polygon?" and then used their algebraic expressions as the basis for their ensuing discussion of and solution to problem #4. (Note: The students had already encountered this problem as problem #1 in the series. At that time ALL students approached problem

#1 by using either an iconic graphic to represent the burrows and tunnels or by means of a chart.)

The collaborative exchange

Following are conversational extracts which have been transcribed from the videotapes. Overlapping conversation appears between /slashes/. While our goal was to include portions of dialogue which clearly convey the meaningful aspects of the exchange, it has been our experience that transcriptions of children's talk from videotapes often appear less meaningful to the reader than to the researchers who had access to both the visual and audio record as well as to the context of the activity. We hope the accompanying descriptions help fill in the gaps.

The pivotal strategy upon which the algebraic expression for problem #3, *Decagon Diagonals* (Zack, 1995) was constructed was as follows: Count the number of diagonals emanating from a vertex, multiply that number by the number of sides, and divide by two. Hence, in a decagon, there are 35 diagonals. Figure 1 below graphically illustrates this representation.

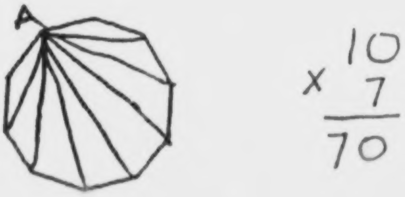

Figure 1. Child's representation of diagonals from one vertex in a decagon.

In their solution to problem #3, Jeff and Micky constructed two variations: Jeff constructed a rule with two components, (S = number of sides):

$$A * S \div 2 = \text{diagonals, where } A = \text{sides} - 3$$

Micky's rule is equivalent but more direct, (Z = number of sides):

$$Z - 3 * Z \div 2 = \text{diagonals}$$

In both solutions it was **Z - 3** or **sides minus 3** because the connections were made to all vertices except for three, namely itself, the vertex to the left and the vertex to the right. The boys agreed on the equivalent nature of these two representations.

Two models of the problem. The boys then went on to tackle problem #4 which required that they generate a number sentence or general rule to determine how many tunnels are needed to connect all 9 burrows.

> J: . . That was just like saying this is what I know, now how am I gonna put it into a sentence? So what I did is I did Point A times sides divided by two then plus sides *'cause you get the diagonals plus the sides, and then that's all the lines you can draw*. (emphasis added)

The model of the problem that Jeff has constructed is a component model in terms of diagonals and sides and this representation explicitly extends the findings from problem #3. He represents it as:

$$\boxed{A * \text{sides} \div 2} + \boxed{\text{sides}} = \text{tunnels}$$

This strategy draws not only on Jeff's mathematics knowledge but also on his understanding of the pragmatic context in which the problem was assigned. As he states, "why would she [Vicki] mention tunnels" if there was no connection. Micky, in contrast did not apply his findings from problem #3 to problem #4 as he was under the impression that a novel solution was required, and he was "looking for something like totally different." Nevertheless, the boys appear to be in agreement at this point:

> M: Okay so it's basically the same thing but you just /add on the sides/.
> J: /Add the sides/.
> M: Except you add it once more without the minus 3.

The **"minus 3"** referred to has been an important element in the boys' understanding of the number of diagonals in a polygon. At this time Micky points out that the sides which are added to the diagonals to determine the total number of tunnels have not been diminished by 3. He goes on to suggest

> M: See, but once you think of it, the Z minus 3 seems pretty weird.

This is the first hint that M's mental model is not J's two-component model but rather the figure as a whole. It seems that if M's model of the problem were in terms of diagonals plus sides, the two components of J's algebraic expression, **(S - 3) * S + 2**, and **+S** might pose no problem. But now the "weirdness" of the minus 3 is introduced, and M goes on to suggest a hypothetical solution which maps to his more holistic representation.

> M: Z times Z.

This is M's attempt to encode his idea that there are connections from one burrow to all the other burrows, or 'points'. The glitch here is that at this point neither boy realizes that 'it', namely the point of origin, does not connect to itself. Jeff completes the representation and Micky asks about the minus 3.

> J: Sides times sides divided by two
>
> M: You're not-, you're not getting back that minus three are you?
>
> J: Ya you are-=
>
> M: When?
>
> J: =You're getting it in a twenty-five 'cause it's times twenty-five times twenty-five divided by two.
>
> M: Ya, will you get that three back?
>
> J: Yes.
>
> M: When?

Genuine inquiry. The question concerning the "3" puzzles Micky, and is asked a total of 14 times throughout the exchange. Interestingly, from a rhetorical and affective perspective, the question is most appropriate and reflects a genuine inquiry pertaining to an important loose end. As such it neither irritates Jeff nor feels repetitive, but rather it drives the remainder of their 20 minute discussion which ultimately results in new knowledge. At this point the boys carry out the calculation which doesn't give them the desired result. This evidence is not lost on Jeff as Micky repeats the question.

> M: =When do we back-, when do we add back that three?
>
> J: We don't.
>
> M: Why not? You have to connect every single line with every single line-burrow.=
>
> J: = No, not in the diagonals.

M: Not in diagonals but I'm talking about tunnels. [leafing through his notebook]

J: Ya I know. Why wouldn't you get it back?

Genuine response. Jeff has gone from "yes you get it back," to "no you don't" and now seems to reflect more carefully on the problem with "Why wouldn't you get it back?" Now it appears that the interest in the question is coming to be shared by both boys.

M: When are you gonna get those back? When you're multiplying or something? You're not-, you're not gonna get them back.

J: (..) That's a good point. [second instance of reflecting on this problem]

J: It does work, but we're not exactly sure when you get the three to connect it back.

M: /Well if we're not too sure about it, / we can't really say it works.

Metacognitive awareness. Clearly from this previous exchange we can see how the boys understand the important difference between knowing that it works and knowing why it works. At the same time they are able to monitor their own problem-solving in terms of those concepts.

Now the explanation is supported with empirical evidence.

J: /I know/ but you don't get the three back because we just tried it out. Twenty-, twenty-five times twenty-five /divided by two-/ [working it out on the calculator]

A number of attempts are undertaken to find the "3" all of which prove to be unsuccessful leaving the boys in the following frame of mind:

M: But when are we gonna get that three back? I'm still wondering. If it has to connect with every other burrow.

J: (...) I have no idea.

M: Well neither do I.=

J: = 'Cause we proved it works / but we don't/ know-

Random hypothesis generation. While the discussion to date has been developed upon some agreed upon principles and has followed each individual boy's conceptualization, it now veers off into random hypothesis generation:

J: Maybe we've lost one number, then when we divide it we gain it back, or multiply it we gain it back?

M: Well how can we divide it and get it back?

J: Well we-, we could multiply it and it goes three over what it should be and then you divide it by two and it evens out.

J: We even it out, by-, we multiply it and it's three too much to equal diagonals, okay? Stay with me here.

M: I'm trying to, believe me, I'm trying to.

Understanding the communicative situation. The "stay with me here" and "believe me I'm trying to" clearly signal the boys' shared understanding of the pragmatic context surrounding their discussion. In this instance they address their communicative roles explicitly as a means of maintaining the focus. This sensitivity to the communicative aspects of constructing a joint-problem-space are revealed on another occasion when M uses a form of direct address to maintain a focus on the problem.

M: /But you want to still/ add back that three. You still want that 3 Jeff.

Four days later: Re-establishing the focus. The same question opens the discussion four days later, but there is a conceptual shift on the part of Jeff which Micky appears to realize. Using drawings in his notebook he demonstrates that it's not 3, and as we see below, Micky replies with "it should be minus one." At this point it is difficult to know if this understanding happened in response to Jeff's explanation or had been developing elsewhere. (For ease of reading, the following portion of the exchange does not include Jeff's speech turns which repeat what he has already stated and overlap completely with Micky's turns.)

M: My only question is where did the three go? Now, that's all I'm wondering about. /I understand/ the rest.

J: /Okay/. I have no idea where the three went. It probab-, but-, the thing is (..) why do you need the three?=

M: Well, but you're-

J: = 'Cause it's not three. It's not three. From here [points to drawing in book] it's three but then you got this point [refers to book drawing]

M: = I know-

M: /But that should actually be *minus one*/= (emphasis added)

M: ='cause it cannot connect with itself, but in the problem it can connect with the others.

Principled exploration. This is both a new and a key idea for understanding the problem. Jeff suggests an exploratory hypothesis which incorporates the new understanding into the previous strategy.

J: I don't know. Try it by subtracting one.

M: /we'll see/

J: /I think you/ wouldn't have-, you could subtract by one and multiply it by sides.

M: Maybe it'll give us uh something.

Test and evaluate. They apply this to solving the problem for a pentagon since they know the answer is 10.

J: Oh okay. So now watch. Okay, it's-, uh let's do this one. We know it's ten. Four times uh five [picking out values on calculator as he speaks. He then looks directly at M and asks] Do you want to do divided by two /or do you want/ to-

M: /Well ya have to-. / Okay, show it now. Twenty, /and that/ would be ten.

J: /Twenty/ Divided by two [looks directly at M]

M: Equals ten.=

J: =Equals ten. So you don't need to add on the sides.

M: [shakes his head] Oh cool.

J: We just found out a new rule.

M: Oh here, wait. We have to try it in like three cases.

J: Ya, we'll try it in three cases, but let me just write it down. Um, S minus one- [writing in his notebook]

They then set about checking their new rule, **$(S - 1) * S + 2$**, in a number of situations.

J: Try it, try it all you want. We've just figured out two ways to figure out tunnels.

Appreciating mathematical elegance. What is striking about the excerpt which follows is that it demonstrates that their appreciation of the new rule goes far beyond just "having another way to solve the problem." The boys characterize the new rule as straightforward and "the simplest way" since it eliminates the addition of the sides, and the subtraction of the 'extra 2.'. At the same time they acknowledge the explanatory power of this rule which enables them to understand that it is 2 that they get back not 3. Overall they both agree that it is better.

M: But that would be the most straightforward, it'll /be the simplest/

J: /That'll/ be the most straightforward because=

M: =You wouldn't have to do an extra, uh, adding on.

J: And an extra subtracting. That's /where you/ get the two back.

M: / ((two extra))/

J: It wasn't three that we were getting back. It was the /two./

M: /Two./ So this is actually better.

J: This is better than before.

The exchange concludes with the boys entering the new rule in their math notebooks. Jeff writes " *best way" next to this new entry, and exultantly says, "Perfect," as he flips his pencil onto the table as a concluding gesture. These gestures in conjunction with the boys' language convey their satisfaction and delight in their accomplishment and suggest an aesthetic appreciation of an elegant solution.

Conclusion

The focus in this paper has been on the way in which genuine inquiry within a collaborative exchange can serve as a mechanism for conceptual change. In examining the exchange between two grade 5 boys collaborating to solve a challenging problem we can see how in order to maintain the focus of the discussion, they drew on their shared knowledge not only of the task, and of specific mathematical concepts but also of the communicative context appropriate for this reasoning activity. The students' sustained search for meaning, their quest for coherence, and their appreciation of an elegant solution in terms of parsimony and explanatory power are behaviors often associated with expert performance (Patel & Groen, 1991; Graves, 1995). We would like to suggest that the source of this performance was the establishment of a meaningful problem which really required a solution and which could be approached jointly. It was M's search for coherence as he sought to account for the 'loose end' of the 3, which drove the inquiry and established the problem within the context of the task itself. While J knew that his rule worked, and he knew why it worked and could explain it, he could not provide a satisfactory answer to M's question. The collaborative construction of the problem space provided the structure within which M's repeated quest for coherence was investigated jointly through the boys' conversational reasoning and which led to new a conceptualization.

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NUMBER PROCESSING:
Qualitative differences in thinking and the role of imagery

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This paper considers imagery associated with children's mental processing of basic number combinations. Children's verbal and written descriptions are used as a means of accessing their imagery and we see how the tendency to concentrate on different objects leads to qualitative differences in imagery and its uses. Children described as 'high achievers' provide evidence of an implicit appreciation of the information compressed into mathematical symbolism. In contrast, 'low achievers' create images strongly associated with visual stimuli suggesting that these children, far from encapsulating arithmetical processes, are mentally imitating them.

INTRODUCTION

> "I find it easier not to do it [simple addition]with my fingers because sometimes I get into a big muddle with them [and] I find it much harder to add up because I am not concentrating on the sum. I am concentrating on getting my fingers right...which takes a while. It can take longer to work out the sum than it does to work out the sum in my head." (Emily, age 9)

Although not explicit in Emily's comment, the meaning associated with her notion of 'concentrate' was related to the mental manipulation of a collection of dots. She was describing the difficulty associated with the simultaneous engagement of external referents–fingers–and the mental scan of a different series of referents–dots. The latter was preferred but the former was used because:

> "If we don't [use our fingers] the teacher is going to think, "why aren't they using their fingers... they are just sitting there thinking'... we are meant to be using our fingers because it is easier... which it is not" (Emily, age 9)

There is no doubt that Emily is only one of many children who prefer to do things 'mentally', or as has been described so frequently by children "in my brain". Many do so because they know things and engage in a form of automatic processing. Others have to make a conscious effort and do so, not because they realise that such effort, with practice, may gradually become automatic, but because of the social environment of the class; "*We are not allowed to use fingers*", "*I am too old for counters" and* perhaps the saddest from a boy of 10 who "*wanted to do things like the clever children*".

Recognising that others do things mentally does not give such children an insight into how things are done by others. This is the focus of this paper. It considers the relationship between procedures, concepts and images in simple arithmetic. To establish the latter it assumes that an image is mediated by a description (Kosslyn, 1980; Pylyshyn, 1973). It builds upon the notion that the language and concrete items associated with objects of thought possess different connotations. These have implications for the quality of children's imagery (Pitta & Gray, submitted) and their processing ability.

The evidence suggests that whilst proceptual thinkers focus on the flexibility of the symbolism and hold symbols as "objects of thought", procedural thinkers may construct and utilise mental images which support their procedural interpretations of symbolism. If it is appropriate, they quickly translate the symbol into another object of thought, for example, finger images, a number track or marbles. It is suggested that mental manipulation with these objects places such strain on the limits of the child's working memory that it impinge against the continuing compression required for "constructive abstraction" (Kamii, 1985) and the development of proceptual thinking.

IMAGERY IN NUMBER PROCESSING

The means through which the co-ordination of actions may become mental operations was of interest to Piaget who believed that new knowledge is constructed by the learner through the use of "active methods" which required that "every new truth to be learned be rediscovered or at least reconstructed by the student" (Piaget, 1976, p. 15). Whether or not *all* children who display competence in the procedural aspects of early number activities undergo this process of constructive abstraction–which Kamii suggests is a construction of the mind rather than something that exists in objects–or indeed whether or not they abstract the appropriate thing is a mute point. The abstraction of a basic counting unit may form a platform from which children may gradually replace slower count-based approaches with more efficient fact retrieval processes. However, such procedural compression may not be so easily achieved by low achievers.

These observations lead us to consider imagery, though, because of the disguised nature of mental images it is only possible to make conjectures about them. They may appear to be well wrapped possessions, covered in many fine layers and sometimes even hidden in discrete packages. We may believe it is possible to shake the package to find out what is inside, but by doing this we run the risk of breaking it. The pitfalls, particularly in terms of operational definitions and interpretation are clearly identified by Pylyshyn (1973).

In cognitive psychology, it has been traditional to characterise mental representations as symbolic: a pattern stored in long term memory which denotes or refers to something outside itself (Vera & Simon, 1994). Such a characterisation is based on the assumption that the knowledge structures possessed by humans are symbolic representations of the world. Images exist, are used and may influence thinking.

It is suggested, though controversially so, that symbolic mental representations divide into analogical and propositional representations–essentially sensory dependent and language like representations. The classical analogical representation is the visual image–though images can be formed from other modalities–which appears to have all of the attributes of actual objects or icons. They take up some form of mental space in the same way that physical objects take up physical space and they can be mentally moved or rotated (see Boden, 1988). Propositions, as mental representations, may represent conceptual objects and relations through, for example, mathematical symbols or spoken words. Gray & Tall (1994) suggest that the symbols of elementary arithmetic serve the ambiguous purpose of representing processes *and* concepts.

Deahenne & Cohen, (1994) suggest that the relationship between different forms of representations may be seen through the presentation and solution of arithmetic facts. Symbolic, verbal and the analogical representations support the transcoding of numbers into whatever internal code is required for the task in hand. It is transcoding approaches which require the use of working memory in the absence of external representations that we are particularly interested in this paper. Symbolism promotes direct verbal routines and flexible transformations by proceptual thinkers. Amongst procedural children, where symbolism is more iconic (static) we see the occurrence of analogical forms of imagery which we suggest may inhibit the potential for flexible interpretation.

METHOD

Twenty four children were selected within in a "typical" school of the English Midlands to represent the chronological ages 8+ to 12+. This provided a sample of six children from each year, three 'low achievers' and three 'high achievers'. Achievement was measured levels obtained in the Standard Assessment Tasks of England and Wales ((SCAA, 1994)) or scores obtained from the Mathematical components of the Richmond Attainment Tests (1974). Children were interviewed individually for half an hour on at least four separate occasions over a period of eight months.

Following the presentation of range of auditory and visual items (Pitta & Gray, submitted) the children were presented with a series of one and two digit addition and subtraction combinations, for example, 6+3, 9–5, 13+5, 15–9. Children's responses were obtained using semi-structured interviews recorded through field notes, audio and video tapes. Children were asked to talk freely about their imagery and what came to mind during the solution processes for each item. Solution approaches were classified similarly to that of Gray & Tall (1994). Whilst external representations were partially identified through children's sensory motor activity, evidence of images relied extensively on verbal and written description by the children. Though no precise claims can be made about the nature of their imagery it is evident that a pattern does emerge.

RESULTS

First and very briefly, because of space limitations, we draw together the general solution strategies and associate these with the type of representations used. The

1. Strategies and Representations: Combinations to Ten

Figure 1 shows the strategies and associated representations used by the low and high achievers to obtain solutions to the number combinations to ten. The representations are subdivided to illustrate percentages which indicate:

- the use of external referents such as fungers.
- where children's verbal description may be associated with conceptual objects

represented by numerical symbols. This notion is loosely tied in with that of propositional images.

- mental imagery associated by anaology with external referents–analogical images.
- the simultaneous engagement of external referents with an anological representation.

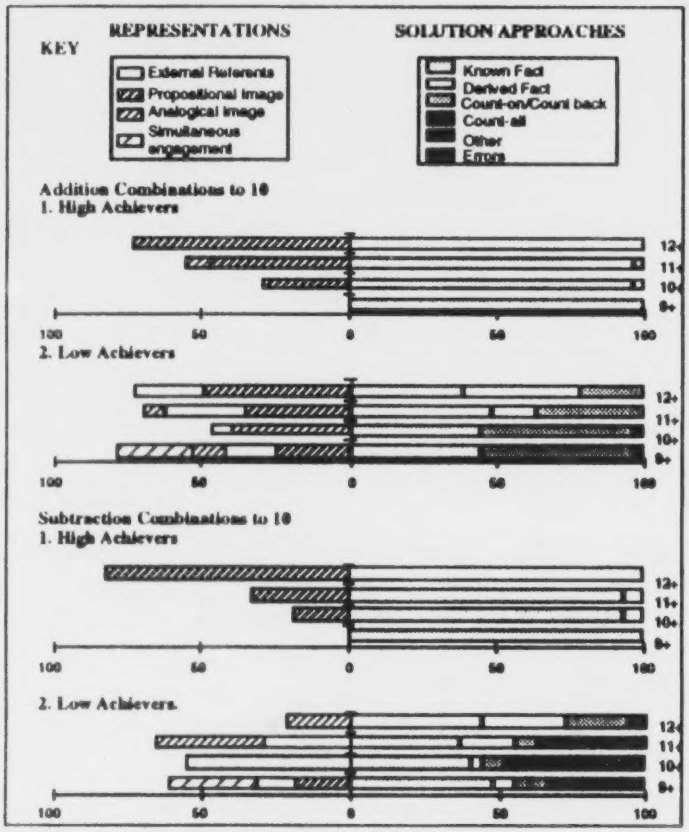

Figure 1: Strategy combinations and representations used to solve addition and subtraction combinations to ten. (Percentages)

Several features emerge from the analysis of Figure 1.

Amongst the high achievers there is the almost complete absence of procedural methods associated with counting and there is no evidence of the use of external representations–verbal enunciation was associated with images of numerical symbols, either the expressions themselves or the final solutions.

Amongst low achievers we note:

- the imagery of 11+ and 12+ children when solving addition combinations is dominated by symbolism supported by analogical representations.
- the absence of symbolic representation amongst these two year groups when dealing with subtraction was associated with the fairly extensive use of external referents by the 11+ group.
- the increasing use of external referents amongst the younger children and, in some instances, we note that these are simultaneously engaged with analogical representations.

At this point, the use of only immediate recall and counting methods amongst the 9+ and 10+ "low achievers" indicates qualities which would enable them to be identified as procedural. The 11+ and 12+, since they collectively display the integrated mixture of known facts, the use of known facts and some evidence of counting procedures may be seen to display proceptual qualities when dealing with addition and subtraction combinations to ten.

Strategies and Representations: Combinations to Twenty

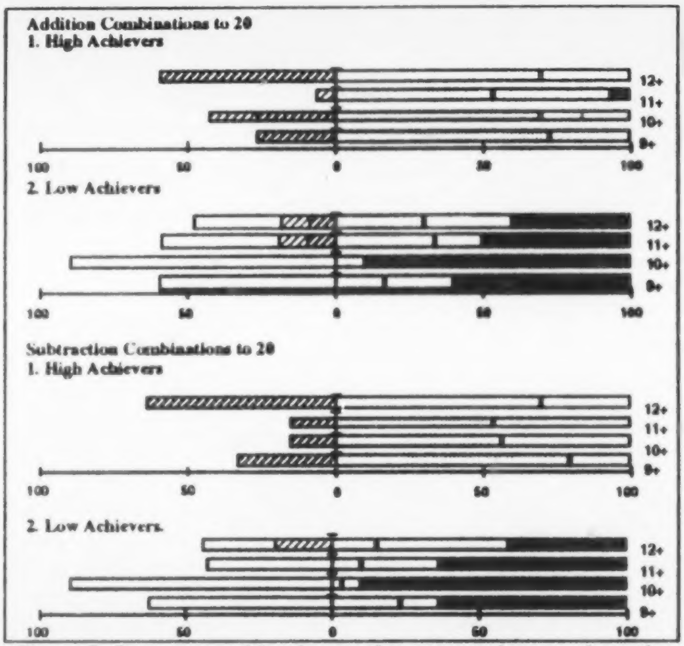

Figure 2: Strategy combinations and representations used to solve addition and subtraction combinations to ten. (Percentages)

The classifications identified for Figure 2 are as those specified for Figure 1. We note immediately the greater proportion of derived facts used by the high achievers and the more extensive use of counting, particularly with external referents, by the low achiever.

The proceptual thinking of the high achievers may be identified through enunciation that refers to images of symbols associated with the initial expressions, semantic transformations of the expressions or from the solutions.

Amongst Low achievers we see that reference to symbolic images is far less evident. The fairly extensive use of derived facts amongst the 11+ and 12+ children is no surprise. Their strategies generally serve to support the evidence given from different samples cited in Gray and Tall (1994).

In general, the evidence shows that the high achievers did not use external representations to solve any of the problems. They either recalled solutions or provide extensive evidence of semantic elaboration, both approaches being associated with "images of arithmetical symbols". Amongst the low-achievers, only the 11+ and 12+ indicate any reference to imagery without the simultaneous engagement of external referents.

Amongst low achievers, we detect a decline in symbolic related imagery and a "regression" from internal to external representations that is both age related and associated with problem difficulty. On the whole their imagery is associated with analogical representations which support counting procedures. We suggest this soon forces them to reach the limits of working memory and makes life so extremely difficult for them that they recognise the "safety" in using external referent. Gear *et al* (1991) have suggested that a component of developmental difficulties in mathematics is a working memory deficit. In our next section we provide an alternative reason which suggests that on the contrary these low achievers may show an extraordinary use of working memory. Their problem is one associated with its use as well as its capacity.

DISCUSSION

The verbal reports of the proceptual children provides evidence of the important role that symbolism plays:

- In those instances where we are able to identify known fact responses, these symbols have skeletal qualities, they carry the ideas and offer the potential for process/concept ambiguity. They require no detail to make them operational. Placed on the minds scratch pad they are interpretations of input data or precursors to verbal output but they are associated with retrieval of simple facts without regard to quantities involved. However, our evidence to date does not allow us to contribute to the controversy that may surround notions of verbal coding (see Dehaene & Cohen, 1994)
- The different degrees of complexity associated with the use of derived facts, particularly with number combinations to 20, provided a variety of examples where expressions were decomposed into simpler known facts, for example, 9+8=8+8+1, 15–9=15–10+1. Perhaps one of the points of interest was the tendency of the 11+ children to indicate that they "did not see anything" although notions of "thought it" were strongly in evidence. This is an issue that we feel needs further clarification. We suggest that nothing was written on the scratch pad and verbal coding could have taken place.

Finding solutions to the expressions through derived facts requires two features not necessarily apparent when using known facts. The first is the possession of a good understanding of the quantities involved in the original problem, for example noticing that 9 is close to 10, and the second involves the use of working memory. However, we suggest that use of the latter is minimised because the children almost intuitively recognise cognitive referents associated with the inputs–disregarding perceptual properties they focus on the relationships associated with the objects of thought–the procept.

It was this ability to recognise the proceptual characteristics of the expressions and their associated symbolism that highlighted the difference between the low achievers and the high achievers. The former had proceptual options available to them but we are not in a position to indicate whether or not their images at this point were functionally significant. The evidence from the low achievers appears to be quite different; no matter what numbers they were dealing with, each individual, on failing to recall a fact, generally they evoked a procedure which they saw common to all combinations. Usually this involved counting, particularly if external referents were used, but this was not always the case when imagery was reported. Usually images given by the low achievers appeared to be functionally significant–they appeared to have a direct role in the processing procedure.

Pitta & Gray, indicate how low achievers interpretations of nouns, icons and symbols were strongly associated with the perceptive aspects of the stimulus. There appeared to be a need to concretise objects. It appears that such distinctive behaviour also guides these children's approach to basic number processing. In the mental world we may see an almost automatic representation of the stimuli as images of countable objects. These may be seen as analogues to, for example, fingers, tally's, number tracks or marbles, each providing an image of the quantity associated with particular numbers. On hearing the expression the children appear to disregard the semantic aspects and move immediately to analogical magnitude representations and use these as anchors for mental manipulation–numbers quickly become concrete objects.

The dominant representations identified amongst the low achievers were associated with a range of images from pictorial representations of a hand with fingers, through iconic representations of fingers and tally lines. The oldest children indicated how they labelled these tally lines and saw images of number tracks or number lines. The evidence was that children who developed such images used discrete objects with a double counting procedure. Two points emerge. First, the horrendous strain on working memory. Not only is the child maintaining sight of the analogical representation but also focusing on discrete numbers in that representation. This is associated with counting-up one set and counting back another. Indeed, one child described how two 'calculators', by description circular number tracks, operated in different ways, one keeping track of how many had been counted by decrementing in ones, the other keeping track of the answer which was incremented in ones. Every calculation, with slight modification, was the same–it always involved double counting. Indeed this was the case with all of the children who used such images–all involved double counting of linearly arranged objects, some labelled some not labelled. Such children seldom gave evidence of the use of derived facts. Indeed it is hypothesised that seeing images of discrete objects supports the counting process but does not lead to the realisation of the power and or compression associated with mathematical symbols. Instead of deriving facts and using what they know about numbers, a sort of vertical processing, the children display some element of creativity in changing their images of countable objects. They use different referents to carry out the same procedure, a form of horizontal processing (Pitta & Gray, submitted).

Such an interrelationship was developed by the few children who used dynamic images composed of marbles or dots. Images of pattern formation dominated their mental manipulation. Marbles can move position, fingers cannot. Fingers require sequential processing, marbles do not.

> "[with] the dots....it's....it's easier because you don't have to keep on thinking, "No its that one I need to move, no its that one or that one", because it doesn't really matter which one you move"
> (Emily, age 9)

But this was not the only advantage, because each item could move position independently of the others. A pattern of ⁙ may easily become ∷• combining readily with ⁖ to make ∷∷, or "two fours". In such a way derived facts may be developed and indeed this did lead to their use amongst two of the low achievers.

Amongst some of the younger low achievers the evidence of simultaneous engagement of mental imagery and external representation caused confusion until one representation dominated over the other. If we do two or more things mentally, for example, count-up, count back and maintain a mental picture we gain some insight into the strains being placed on working memory.

CONCLUSION

There are limits to the size of working memory. Whether or not these limits are different for those children we identify as high achievers compared to those we see as low achievers is not resolved. Their implicit appreciation of the information compressed into numerical symbolism enables them to focus on the detail appropriate at the moment. However, this feature is not unique to their approach in mathematics. In the broader context symbols, and the ability to focus on the many relationships asssociated with them, provides them with an economical means of utilising the power and space they have available. We would not like to give the impression that high achievers did not use and manipulate visual images. When dealing with more difficult two digit combinations all high achievers considered visual symbolic images in vertical form, even though they were given verbally, and made transformations which enabled them to process them more easily. Low achievers, giving more attention to different elements, found it even more difficult to mentally hold the initial inputs. They appear to place much greater reliance on a visual stimulus and create and manipulate images associated with this. They have a much greater tendency to talk about things that may be captured by the senses and their imagery tends to be strongly associated with real concrete objects.

Notions of procedural encapsulation and the steady compression of lengthy counting procedures into numerical concepts imply that children recognise links between inputs and outputs. It would seem that far from encapsulating arithmetical processes some children reconstruct these processes mentally. Attempting to match their thoughts to given representations may only help them see things enactively, as with marbles, or iconically, as with the number line. It is those who realise that representations may be used to simplify ideas and are not intended to stand alone who will share in the construction of meaning.

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IDENTIFICATION OF VAN HIELE LEVELS OF REASONING IN THREE-DIMENSIONAL GEOMETRY[1].

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ABSTRACT: *An analytical study of the behaviour of third year Teacher Training College students when carrying out tasks, designed on the basis of the Van Hiele model, to solve problems on solids is the foundation of the characterization we here propound for the levels 1, 2 and 3 in the field of the three-dimensional geomètry. On detailing our proposals we have also taken into account the characteristics already established as a result of research in this field and those of Van Hiele levels generally.*

RESUMEN: *Un análisis del comportamiento de los estudiantes de 3º de Magisterio cuando resuelven actividades sobre sólidos, diseñadas en base al modelo de Van Hiele, es la base para las caracterizaciones que proponemos para los niveles 1, 2 y 3 de Van Hiele en el campo de la geometría tridimensional. Para la elaboración de esta propuesta también hemos tenido en cuenta las características ya especificadas en la investigación realizada para esta área y las características generales de los niveles.*

INTRODUCTION

The Van Hiele model of reasoning in plane geometry and other areas of mathematics has been the subject of considerable and important research the world over. It has been demonstrated that the characteristics of different areas (arithmetic, algebra, geometry, etc.), reveal marked differences in the kind of reasoning students employ.

As regards 3-dimensional Geometry there has been little research, but since the Van Hiele model is based on the experience of its authors as geometry teachers, it may well be especially suitable for this area of mathematics. There have been several isolated approaches to 3-dimensional Geometry based on the Van Hiele model. Some attempt to formulate specific characteristics for Van Hiele levels as applied to solid geometry have been made in Hoffer (1981), Lunkenbein (1983a), (1983b), (1984), Gutiérrez and others (1991), Pegg, Davey (1991), Davey, Holliday (1992), Gutiérrez (1992). But as Gutiérrez (1992) indicates, the characteristics prescribed for the different levels of reasoning are insufficient. More research is needed. On the one hand, the levels of reasoning applied in practice in the case of space geometry must be specified; on the other hand, the practical exercises needed to enable students to move from one level to the next must be designed, taking into account the phases propounded in the model.

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In this paper we set out the characteristics we propose for Van Hiele levels 1, 2 and 3 in tridimensional geometry[2]. Our proposal is based on research in which we have been involved, using third year Teacher Training College students, into the design of practical tasks with solids, designed on the basis of the Van Hiele model[3].

THE VAN HIELE LEVELS FOR SOLID GEOMETRY

The aims of our research were to obtain operative and detailed characterizations of each Van Hiele level in terms of the students' behaviour in their work with solid geometry, and to enhance their level of reasoning.

In order to define the characteristics of the different levels of reasoning we used several sources. We considered descriptors specified in research relevant to plane Geometry (for example, Burger and others (1986) or Fuys and others (1988) provide accurare descriptors) and the characteristics of these levels formulated in three-dimensional geometry research. We also analyzed the answers of students to specific problem tasks given them to work on at home on the day before those tasks were discussed in class. Their answers were collected before the discussion began. We then noted the questions raised by the students in class, and their answers to problem tasks they were subsequently given to solve.

We summarize below the characteristics we identified for level 1 and level 3, and focus in detail on the descriptors found for level 2. Where the ability in task may correspond either to level 2 or level 3 (depending on the kind of property, relationship, or families of solids under consideration) we will indicate it.

Level 1 (Recognition)

At this level students deal only with visual information. They can perform tasks dealing with recognizing, naming and building some three-dimensional objects of different sizes which may be presented from different distances. In addition, nets of some simple solids can be constructed, dismantling models of solids. Students can change the form of some solids by making cuts in concrete models, and identifying the solids obtained. They can also describe a solid by reference to its physical aspect or from prototype examples taken from their physical environment. Moreover, they can compare or classify the solids on the basis of global physical similarities or differences between them and establish dichotomic classifications and classify only those figures with which are most familiar. To describe a family of solids they merely choose a familiar example of the family.

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[2] Some authors number the Van Hiele Levels from level 0; in this paper they are numbered as follows: Level 1 (Recognition), level 2 (Analysis), level 3 (Informal deduction).

[3] The results presented in this paper are part of the author's project of doctoral thesis.

At this level, students may, in their answers, using terminology or refer to geometric properties incorrectly, imprecisely or inadequately, and it is not on those terms that the answers will be based. Answers of this sort given to questions on solid geometry, may reflect the students' previous experience of, or contact with the study of plane geometry.

Level 2 (Analysis)

At this level students begin to recognize that objects have mathematical properties, even though their thinking is still based on physical perception. They can by experiment establish relations between the components of a figure and between several or different figures. Tasks dealing with different abilities can be performed, such as the following:

1) To Identify a solid as an example or non-example of a family of solids. Students may base the answers on their own definitions (being lists of properties) and not taking into account definitions given by the text book (or the teacher). The model, presented in different positions, can be identified adequately whether it is presented as material model of the solid or its structure. Adequate identification is also possible when the models are presented as physical objects, as pieces of a game, inmersed in a structure, or in a puzzle.

Certain relations between given models can be understood: some can be recognized as aggregates of others. Models can be separated and the elements of the resulting models can be identified. Relations can be established between the elements of one family and those of others from which it has been derived. For example, it can be observed that if 4 space diagonals are made in a cube, it is divided into 6 equal pyramids, whose bases are the faces of the cube, whose height is half the edges of the cube, and whose lateral edges are half the space diagonal of the cube.

2) A solid can be described on the basis of its geometric properties. These are determined by observation, measurement, drawing and construction of models. Students at this level can also enumerate properties for a family of solids or for a general case (for example, a prism n-gonal), starting experimentally and generalising the properties from some examples. They can already grasp that a mere example does not replace a family, and that to describe families of solids one must seek several different examples and draw general conclusions from them.

However at level 2, students are not yet able to determine, as critical attributes of a family, properties that contain terms such as "as much", "as a minimum", "at least" "as many... as", "the same ... as", "two different types of faces". They cannot use these terms to reformulate properties so that these become properties of a family that includes an other. At this level the list of properties indicated for families that contain other more specific families tend to leave out specific examples. For example, if students are asked to state a property of right prisms with a regular base in terms of different measurements for the edges, they leave out examples which also belong to the family of prisms with regular faces. They will say that "The right prisms with a regular base have edges of two different lengths".

3) Examples of a given family of solids can be constructed up with various commercial materials. Different nets of a solid can also be built. Students can make structured analysis of the models by levels, or separate a model in layers, or observe the faces bordering a given face, or those which meet in a vertex. All these observations can be applied to find nets of the solids.

4) Students can identify mathematical information provided by a solid model or a drawing of it, explain their answer in terms of properties, or apply this information to one of the nets.

5) Students can tackle problems on classification as the following: Establishing classifications-partitions, based on geometric properties when the criteria have a strongly visual component; naming the established families; identifying models of solids as examples or non-examples of subfamilies; listing the properties of established families; specifying all the types of example of a given family which satisfies certain conditions. For example, given a set of solid models, students can select examples of parallelepiped that they are non-examples of orthohedra.

Faced with the problem of classification at level 2, students can also choose appropriate examples and non-examples to show whether that certain statements interrelating families of solids are, or are not, correct. They can decide whether a given relationship between families of solids is correct unless the relationship is stated in terms of "There cannot be ... that are not... ". In this case, to understand these terms and to determine what has to be proved requires a level 3 of reasoning. Students can make statements using the expresions "always", "sometimes" or "never" in order to show if between two given families of solids exists a relation of inclusion, if they have common elements, or if they are exclusive. A tree or net diagram can be constructed showing the relations between certain families.

For families with a marked visual component, or with which students are very familiar, the relation of inclusion or exclusion can be established and substantiated by proving that the properties which one family (or its definition) exhibits are also exhibited by the other, or that no example of one family exhibits the properties of the other. For example, it can be proved that cubes are always prisms by showing that they satisfy the definition of a prism. It can also be proved that the oblique and right prisms are disjoint families by checking that any example of a right prism is not an oblique prism. But in level 2 the students cannot reason in a mathematically complete way, to prove, for example, that prisms with regular faces are convex or that pyramids with regular faces never are archimedean polyhedra.

It may be observed that at this level, the students do not admit the inclusion of classes between given families of solids if it has not been previously considered as inclusion in terms of examples. Thus, even though it can be observed that properties of parallelepipeds (for example that opposite faces are equal and parallel) are also satisfied by orthohedra, it cannot be deduced from this that an orthohedron is a parallelepiped. This relation could be verbalized if previously examples of orthohedra have been included as examples of parallelepiped.

6) To associate properties to given families of polyhedra and to identify families of solids from one o more given properties.

If the properties contain terms such as "as much", "as a minimum", "as maximum" or "at least", students interpret them as "exactly". This interpretation is not mathematically correct. The term "different" that appears in "as many different measurements as...", "the same different measurements that..." is interpreted as "which has to have different elements". "Faces of the same type" tends to be identified with "That are equal"

Properties which present a further difficulty for students reasoning at level 2, and also unable to give mathematically correct answers, are those which have not been ascertained by experiment and whose correct mathematical verification requires deductions, or taking into account several elements of different types. For example, students cannot prove that the family of prisms with regular faces verifies the following property: "The number of different measurements for the space diagonals is equal to the number of face diagonals + 1". We can also include in this group, the properties that contain the term "exactly" and which oblige students to select families or very specific elements which are possible solutions, taking as a starting point numerical data (for the edges or the different measures of the face angles). These properties lead to a problem of proof, because all the possible solutions that satisfy the property must be listed and it must be proved that there can be no other solutions.

7) To evaluate sufficiently and to explain the answers correctly, giving examples or non-examples, that given two families, one of which is introduced by a definition, are not related by inclusion. For example, when the family of the deltahedra is introduced with a definition, students can give an example and a non-example showing that pyramids with regular faces are sometimes deltahedra.

8) When the inclusion between families has a marked visual component, or is an inclusion relationship that is usually considered by students in terms of examples, the relationship between two families can be used to establish properties of a family, considered as properties of the other family. At the end of the teaching

course unit, students who reasoned in level 2 can use the relations of inclusion that exist between the cube, orthohedron, rhombohedron, parallelepiped and prism. They can also apply the relations of inclusion between the family of prisms and the various families, already identified in that family, whose names contain the word "prism". For example, students can understand that the rhombohedra satisfy the properties of parallelepipeds; or that prisms with regular faces satisfy the properties of prisms.

However to determine adequately all the families under consideration (or all the families comprised in such families) and to describe correctly the interrelation of families in terms of properties, level 3 of reasoning is required. For example, at level 2, students cannot demonstrate adequately that rhombohedra satisfy the properties of quadrilateral prisms, of convex prisms, and of parallelepipeds. Nor can they explain correctly that the properties of the first two families belong also to the properties of the last family.

9) To produce formulae which give the number of faces, vertices, edges, or a given sort of angles (face angles, dihedral angles and vertex angles), for a given family of solids (prisms, antiprisms, pyramids and bipyramids), and to apply those formulae for a particular value of n. Furthermore students can justify a formula either by generalizing for n the results obtained from specific examples, or by counting the elements in a structured way (for example, separating a polyhedron by levels in order to count its vertices) and making a generalization for each level. For example, the antiprism is seen as a closed band of 2n triangles plus two polygons, so the number of faces is, $F= 1 + 2n + 1= 2n + 2$ and the number of edges is $E= n$ (of a base) + 2n (those in the band of triangles) + n (of the other base)= 4n.

However, to determine formula like that giving the number of face diagonals or space diagonals for a given family, and to justify the results, requires level 3 of reasoning.

10) For very specific families, students can check, by counting, measuring, or applying already known results, formulae that give the number of certain elements or their measurement. For example, they can verify that, in a hexagonal right prism with a regular base, the sum of the angles of the vertices is $12(180 + 120)$.

However at this level students cannot prove in a mathematically correct way that this result is valid for any hexagonal prism. In level 2 this result can be justified only for right prisms with a regular bases.

Level 3 (Informal deduction)

At this level students begin to develop a capacity for rigorous reasoning and are able to handle the simplest elements of the formal system (definitions and implications in a single step). Logical classifications of the solids (inclusives-exclusives) can be made, based on properties or relationships already known, formulated with mathematical accuracy. Students can work through and solve adequately the problems of classification that arise, with the properties, definitions, or relationships between families given at that moment. They can grasp the need for definitions and why families of solids must be defined in a formal way. Furthermore, they can understand the requirements of a correct definition and succeed in fomalizing it. Various propositions can be proved in a informal way. Deductive methods are presented together with experiments, which will therefore allow students to deduce properties taking, as a starting point, other properties which had previously been obtained experimentally.

Let us now specify some of the abilities at this level of reasoning: Students can conceive the examples as representative of classes, and are able to choose them in such a way that the answer is mathematically correct. They can understand that the properties or definitions given for a family, or the diagrams given to represent the relationships between families, reflect the type of classification (inclusive-exclusive) that is established. They can understand the logic quantifiers. They can list properties in which one must take into account several families (because either common properties or properties of a family that are not satisfied by others are considered) and apply relationships of inclusion between families to simplify a given task.

At this level, the meaning of deduction is not yet realized, nor is the structure of a proof understood. Students can understand a proof explained by the teacher, but they are not able to produce it by themselves. They cannot distinguish an implication ($p \Rightarrow q$) from its reciprocal ($q \Rightarrow p$). They are unable as yet to understand the function of axioms or the logical connection between statements, or the axiomatic structure of mathematics.

DIRECTIONS FOR FUTURE RESEARCH

As regards the characterization of the levels for space geometry, once the characteristics of Van Hiele levels 1, 2 and 3 (which are the levels of reasoning applicable to the students who have participated in our research) have been specified, it is necessary to identify the descriptors for level 4. We shall be carrying out research with students from the Faculty of Mathematics, where they are expected to be able to achieve a mastery of level 4.

On the other hand, once the descriptors of Van Hiele levels have been identified in the field of the 3-dimensional Geometry, the research into the Van Hiele model can be continued by working on the assessment of the development of the level of reasoning of students in this field. We are investigating it with third year teacher training school, but presenting the results obtained is beyond the scope of this paper.

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WORKING WITH 'THE DISCIPLINE OF NOTICING': AN AUTHENTICATING EXPERIENCE

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In August 1994 at PME XVII in Lisbon John Mason gave a plenary address in which he presented his development of a research methodology appropriate for practitioner research. This methodology he has variously described as 'Noticing' and 'Researching from the Inside' [Mason 1993, Mason 1994a, Mason 1994b].

This is a report of our use[1]*, as practitioners in mathematics education, of such an enquiry research paradigm – with our struggles of how and where to start, with methodology, with discipline and with working ourselves and with teachers on researching into our classrooms.*

We have written this report in three major sections each of which reflects concerns that we encountered during our research. In our concluding remarks we attempt to give a personal overview of the effects of this experience on us as researchers and practitioners.

What is data for a practitioner researcher ?

This question emerged for us very early in our enquiry. A, too obvious, answer might be, our experience. It seemed to us that there are problems with this.

Much writing by our students and much talk by teachers is hard to enter. It is characterised by highly generalised anecdotal narratives and sweeping value judgements. We find it difficult to get students to reflect upon their experience. We wish to make the assertion here that within their narratives there was nothing to be examined, nothing to be read. They cannot re-read their accounts; but then in order to re-read there needs to be something to read. They have created no-thing.

An immediate issue for us then was the question "What is going to be our data?"

In other fields it is not such an important question because of their traditions. In Cultural Studies, Media Studies and Literature the data which is examined is clear. It is posters, films, adverts, poems. They are then read using the tools developed by writers such as Barthes, Lacan, Foucault [see, for example Easthope 1988].

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[1] This paper is based on our work together with colleague Una Hanley See Hardy Hanley Wilson 1994

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It seemed to us that Foucault and Barthes have changed our notion of what can be read, what can be opened up to interpretation. Foucault's examinations, in the Archaeology of Knowledge, have produced the data sets called 'discourses' which are the ways in which the medical profession for example have by talking, by theorising, produced 'the insane', 'the sick' and so on. Within education, writers, such as the psychologist Valerie Walkerdine, have used his methods, amongst others, to examine critically the discourse of developmental psychology and its production of 'child' as a consequence. Posters, and films are artefacts and, as such, are already out there, exterior to ourselves, available for examination. Valerie Walkerdine examined the writings of, for example, Hughes and Tizard and re-read them, offering re-readings of their data which consisted of transcripts of dialogues between children, parents and teachers.

The notion with practitioner research, of teachers and student teachers reflecting on their own practice, poses problems in this respect. Within this framework what is it that can be held up, exteriorised, for examination, for reading and re-reading by practitioners?

We decided to work with a methodology which has been described and developed by John Mason of the OU, which he terms 'Noticing'. This involves a disciplined way of working with other teachers in the telling of fragments from their experience. Particular exercises are offered to enable teachers to work with this. The intention is to be able to turn **unexaminable experience**, in the sense we have described, into **enterable moments**, recognisable by colleagues as resonating with their experiences. It is about articulating and symbolising experience, and by working with that articulation.

Gattegno has written;

> "The main difference between the existing, recognised sciences and all the accumulated experience of millennia is that the first have been codified socially and given a status by their journals, their academies, their annual or regular congresses, while the other is hanging in an untouched universe which the future may want to reach and explore."

There is a philosophical turn here, a certain distancing of ourselves from what we habitually do as practitiones and say as we describe our practice; the creation and insertion of a gap between our experience and our reflection based on what we say.

A coach told a basketball player who was practising shooting to "jump up high, hang there, make your shot and then come down". Hanging there can be seen as analogous to inserting a gap, to the creation of distance. Caught up in the momentum of our experience it is impossible to notice and to choose, in the same way as the shooter, caught up in the momentum of jumping up and down, does not have the space to aim and to shoot.

Certainly it is possible, at this stage, to question the validity of the data we generate and the 'truth' of our readings and rereadings. We return to this question later.

Exercises with anecdotes

We need to consider the issue of turning theory into practice; to find ways that assist practitioners in the collection of anecdotes which capture moments from within their experience and invite them to work further on these in such a way as to create usable data. This identifies two roles, a teller of an anecdote and a listener to anecdotes.

The teller and the listener are both in the process of reading and validating the story. They need to recognise that both are talking about the same thing. Hence the need to work on the focussing of the anecdote in such a way that this recognition can take place. Experience tells us that this recognition is not easily achieved, as anecdotes tend to be emotionally charged, lengthy and difficult to enter.

For an anecdote to be turned into data it needs to be enterable by both the teller and the listener. The teller needs to be able to review the anecdote, the listener seeks resonance with her own experience. The account must be focussed, concentrating on few points rather than many in the first instance. One way of working on this is to limit the time available so that the teller becomes involved in some form of personal editing process or, alternatively, both can focus on an aspect of the story that has been identified as potentially fruitful.

Using the vocabulary of 'Noticing' these potentially fruitful moments might be described as *salient* or *a moment of energy*. The teller needs to tell the anecdote as *briefly* and as *vividly* as possible. Stripping away the impenetrable overlays is a difficult process. The listener needs to assist the teller in identifying where the emotional energy lies. Tellers of anecdotes need to recognise when they were offering *an account for* a situation rather than giving *an account of it* and resist the temptation to justify and explain away responses to situations rather than focussing on the response itself.

The following is an extract of an account by a primary school teacher. She has produced a brief and vivid account of a particular experience with children and then gone on to offer a commentary on parts of the interaction. Finally she points up the significance of the incident for her.

An extract from A's journal

> The worksheet showed eleven "parcels" which had to fit a "tray". One pair were working on this.

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Pupil 1: I have only ten parcels, there should be eleven.
I must have missed one.

Teacher: Why don't you ... [A pause – I was going to say "...
go through them and check each one" instead –] ...
you need eleven to do the investigation.

[At this point I walked away but continued to observe.]

Pupil 2: Have you got the same as me?

Pupil 1: No, we had better match them.

[They then started to place the "parcels" on the sheet of A4. How I wanted to intervene!]

I made the decision (almost too late!!) to leave them with the problem. Even now when I re-read the extract I can recall the feeling of being awake to the moment of decision and being able to choose an alternative to what may have been an automatic response.

Symbiosis

We have flagged up our concern with issues to do with the nature and role of data. We want now to consider the relation of data and data collection to the enquiry process as a whole. We want to show up threads inherent in our discussion of data that identify two particular elements within the enquiry: the element that is to do with the generating and identifying of a focus of enquiry, of strands running through experience; and also the element to do with the interpreting and validating of those strands.

For us there is a sense that these elements of the enquiry are caught up with each other, that in the identification of relevant data we cannot help but reflect simultaneously on that data and consider its relation to other data.This data is compared to existing strands in our experience, and we notice jarring and resonance with previous interpretations. New interpretations start to form.

In this section we want to explore this symbiotic relationship. The 3 elements; **Data,** Identification of **Focus** and **Validation** form the nodal points of the diagram we build up below.

We start to create an image of this symbiosis, not at any asserted beginning, but by considering the exercise of capturing moments of energy from one's experience, in the form of anecdotes. These may be identified by noticing moments of tension, resonance, jarring; moments that present themselves as salient in some way; these could be events in which one is aware of making a decision in that moment.

> 'Certain aspects of an event or situation stand out and are attended to while other details are not even noticed. The aspects of an event or situation which make it stand out are principally aspects resonant or dissonant with past experience or present' (John Mason)

We assert that these critical incidents are the most important to consider, although they may often be the moments that we habitually step round, avoid or choose not to see.

In following this exercise through, such anecdotes can be worked on, by telling and retelling, by reading and rereading, in order to make them enterable. This involves recapturing the incident and reworking the anecdotes, in writing or group telling, so that salient aspects of the incident are articulated and described.The intention is that the incident becomes re-cognisable to others, so that there is resonance for others with their own experiences. This requires teasing out moments of emotion, stripping away accounts for the actions and reactions described and working towards an enterable account of the incident.

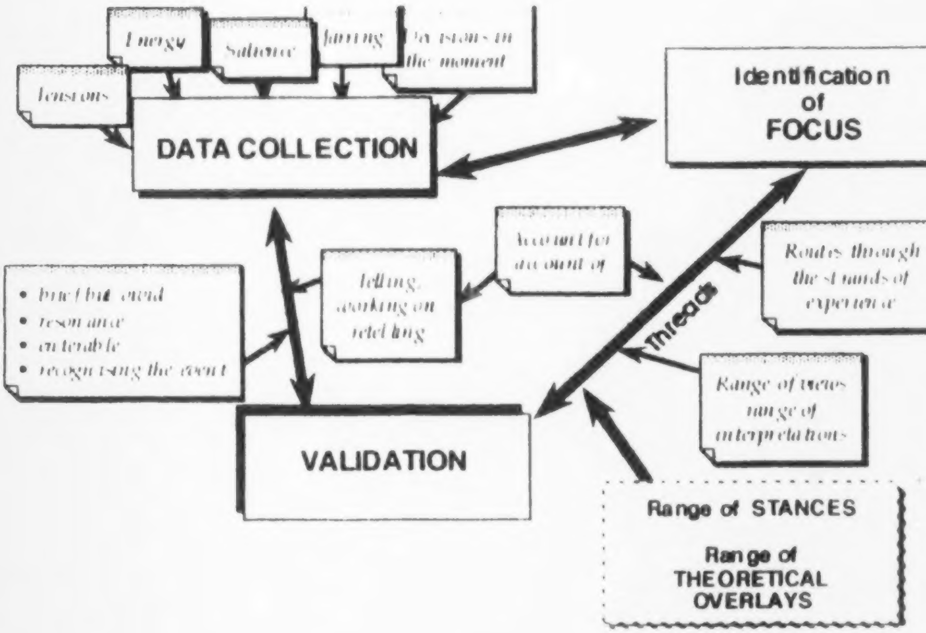
One possible consequence of this exercise is the **validation** of the importance and significance of that data for those working on the anecdote. If there is resonance with and recognition of the incident in others' responses, this in itself constitutes a form of validation. The resonance and recognition of the incident with the teller's own previous experience serves as another act of validation.

And **importantly**, this may form part of the constitution of another element of the symbiosis, **the identification of focus**. The re-cognition of a salient moment may mark a move towards the identification of a focus which provides a useful entry into a view of practice. As importantly, the moment needs to be reviewed in as much detail as possible by both speaker and listener in order to identify patterns of preoccupation and interest.

Through the telling and reworking of these anecdotes it is possible to see **strands within one's own experience**, routes through one's unexamined practice, that had been previously unilluminated, silhouetted amongst previously unrelated threads.

Working on anecdotes by group telling or reading, producing and sharing enterable accounts of incidents, simultaneously constitutes another element of the enquiry – the awareness of a **range of interpretations** and views – and offers stronger validation of interpretations and further cohesion of a strand of concern or awareness. In this process more data becomes available for consideration. A form of co-generation.

'and the one doesn't stir without the other' [Irigaray]


Energy
Salience
Jarring
Decisions in the moment
Tensions
DATA COLLECTION
Identification of FOCUS
Account for account of
Telling, working on retelling
brief but vivid
resonance
relatable
recognising the event
Routes through the strands of experience
Threads
Range of views range of interpretations
VALIDATION
Range of STANCES
Range of THEORETICAL OVERLAYS

So the three elements of data collection, identification of focus, and validation coalesce further within this picture each taking a multipurpose role.

To return to the concern of this section, we have shown that the identification of a focus is necessarily in a symbiotic relationship with the creation and role played by the data, and its validation. This is not undisciplined.

Concluding Remarks

Last year we not only attempted to use "The Discipline of Noticing as a framework on a Masters Unit but to work with it ourselves. In particular we worked upon the creation of brief-but-vivid accounts.

This was, in part, an attempt to test out, by working on ourselves as well as with others, the conjectures that their creation is both possible and worthwhile. Certainly it required effort and practice and in that sense was not unproblematic. What we observed was that early talk and writing by both students and ourselves was characterised by containing highly generalised anecdotal narratives, both about practice and about pupils. We have suggested that within these narratives there was nothing to be examined, in the sense that nothing salient, no moment of energy was identifiable. This energy, we have found, signifies something worth examining further. The exercise seemed less about giving "a proxy a matching experience" or communicating a "shared meaning", but more one of creating data, getting a sense of what there is to be studied and reflected upon.

The issue of validity was to do with the identification of phenomena which were recognised as part of the participant's practice worth reflecting upon. The telling, and re-telling up until recognition or resonance provided data for reading and re-reading.

As we said earlier, in the cultural studies field there are artefacts exterior to ourselves available for reading. Valerie Walkerdine examined the writings of Hughes and Tizard and offered re-readings of their transcripts. A practitioner researcher needs to construct the data from their own experience which is exteriorised and held up for examination. The validity of data we generated about our teaching and the 'truth' of our readings may be questioned.

There has been a debate going on within psychoanalysis[2] about the validity of the case studies produced by influential practitioners, including Freud. Freud concealed the provenance of his story of an infant saying "fort – da" (here – there) while playing with a cotton reel. Stekel, when taken to task by Freud for revealing the names of his clients at a conference is supposed to have replied that he had not only made up the names, but also the stories. Kohut's case studies are thought to have been based upon his own self-analysis.

How much does this matter? We want to suggest, that from the viewpoint of a reflective practitioner, the issue of validity is much more one of whether the retold anecdotes are recognisable by other practitioners and are so discussable than of whether they are objectively true or not. The anecdotes come with their own truth, in a way similar to that which Flaco Jimenez, as Ry Cooder said[3], 'brings his own authenticity on stage with him'.

The awareness that we were not dealing with a cyclical procedure was reinforced by our experience of working with teachers on this Masters Unit. Within their enquiries the teachers were involved in creating data, generating and identifying a focus of enquiry and interpreting and validating those strands. However these elements were not engaged with in a linear order – cyclical or not. We have earlier indicated that in the identification of data we are simultaneously reflecting upon this data and its relation to other data and in that sense the elements of the enquiry process are inextricably caught up with each other.

In this paper we have attempted to create a structural form to represent the simultaneity and symbiosis inherent in this process and capture this diagrammatically. This image has no asserted beginning, no end. We are 'always in the middle, between things, interbeing, *intermezzo*'[4] . This allows us to suggest that the identification of focus is necessarily in a symbiotic relationship with the creation and role played by data, and its validation. The process of 'Noticing' is complex yet disciplined.

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[2] Recent discussion of this issue emerged on a psychoanalysis email discussion network

[3] From BBC 2 Arena transmission on Ry Cooder's music

[4] This is taken from a description of rhizomic thinking a phrase used by Deleuze and Guattari to refer to non-linear activity

Of course, we come to the enquiry process with existing interests and concerns, and there is always the question of whether we are merely reinforcing current concerns (or even fetishes) – the obsessive 'seeing 7s everywhere' – but we find that we can insert a gap and hold these at a distance and conduct a valid enquiry.

In looking for ways to work with our Masters students and also to support our own research we found that other available frameworks [see dicussion in Mason 1994 pages 52-4] seemed neither to give explicit recognition of our personal experience as teachers, nor to offer us an authentic description of how we had come to our professional knowledge. We chose to engage with the Noticings – Researching from the Inside framework as it supported this experience, whilst making our reflections more systematic.

Here we have tried to speak validly with a voice from our own practice.

We have both recently been involved in practitioner research, including that leading to post-graduate qualifications. We feel that 'Researching form the Inside' has provided us with some conceptual tools and a framework that have enabled us to also speak validly there with a voice from our own practice.

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CLASSIFYING PROCESSES OF PROVING

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This paper outlines a preliminary classification of the kinds of justifications that students offer in mathematical contexts, i.e., their "proof schemes." The classification is based primarily on the work of students during teaching experiments and individual interviews, with secondary and post-secondary students. The dominant, natural proof schemes of most students--even university mathematics majors--are not ones accepted in the mathematical community as giving mathematical proofs. Transformational proof schemes are viewed as essential for advancing beyond these schemes; teaching experiments with university students suggest that many students can make pleasing progress toward expecting and giving acceptable mathematical proofs.

Many researchers have given attention to different aspects of the learning and teaching of proof (e.g., Bell, 1976; Chazan, 1993; Fischbein and Kedem, 1982; Hanna, 1990; Martin and Harel, 1989; Senk, 1985; Yerushalmy, 1993). These indicate that the ideas of proof are difficult for students to learn, at least as they are currently taught. A quote from Poincaré summarizes our position toward the teaching and learning of proof in mathematics:

> It is difficult for a teacher to teach something which does not satisfy him entirely, but the satisfaction of the teacher is not the unique goal of teaching; *one has at first to take care of what is the mind of the student* and what one wants it to become. [via Artigue, 1994; emphasis added]

Accordingly, we have been concerned with attempting to determine what *is* in the minds of students, when proof comes up in mathematics. Others have had the same concern. For example, Chazan (1993) noted that U.S. high school geometry students were skeptical that a deductive proof assured that there were no counterexamples to the assertion proved, and that a proof was only further evidence that a conjecture is true. Fischbein and Kedem (1982) found that among students in an Israeli program of studies involving the greatest concentration on mathematics only about one-third of the students who had endorsed a statement and its proof realized that further checks of specific instances would be superfluous.

Our approach has been to focus on justifications, and to view a mathematical proof as the type of justification that is usually accepted by the mathematical community. During interviews, mostly of university students in courses for mathematics majors, we have attempted to determine what sorts of justifications convince them, and what sorts of justifications they would offer in order to convince others. During teaching experiments with university students, the thrust has been to help students refine their own ideas about what constitutes justification in mathematics.

Categories of Proof Schemes

The notion of "proof scheme" has been useful to us. Proving (or justifying) a statement includes two aspects: ascertaining (convincing oneself) and persuading (convincing others). An individual's proof scheme consists of whatever constitutes ascertaining and persuading for that person. Hence, a proof scheme is idiosyncratic and naturally can vary from time to time and from context to context, even within mathematics. It is important to note that "proof" as used in "proof scheme" need not connote "mathematical proof." The teaching experiments have had the intent to identify and alter students' proof schemes, and the interviews to test the sufficiency of the classification. The categories as currently conceived fall into three major classes. In a few cases the labels for the proof schemes are tentative, so the reader should rely not so much on the labels as on the brief descriptions and illustrations.

The External Conviction Proof Schemes

The earmark of the external conviction proof schemes is that justifications hinge on such external features as the endorsement of an authority (the authoritarian proof scheme), the form of the argument (the ritual proof scheme), or meaningless manipulations of symbols (the symbolic proof scheme).

The Authoritarian Proof Scheme. When students are not concerned with the question of the burden of proof, and their main source of conviction is a statement given in a textbook, uttered by a teacher, or offered by a knowledgeable classmate, they are exhibiting the authoritarian proof scheme. When asked how they might convince someone of a particular result, statements like "I would try to find it in a book" or "I think my professor said it, so it should be in my notes" would be offered under this proof scheme. The value of proofs may even be questioned, perhaps because in so much of the mathematics that the student has experienced the emphasis has been on the results, with little or passing attention to the reasoning processes used to arrive at those results. In the teaching experiments, where "why" is a routine expectation as well as "how," students have gradually become less unquestioningly accepting of assertions deliberately made by the instructor to test their willingness to accept the mere word of the "authority."

All this is not to say that accepting the word of an authority is all bad, of course. Even noted mathematicians are no doubt on many occasions willing to accept a result without examining the details of a proof. Rather, it is the attitude of helplessness in the absence of an authority, or the view that justifications are valueless, that handicap the students with an authoritarian proof scheme.

The Ritual Proof Scheme. Martin and Harel (1989) examined whether students' judgments of an argument are influenced by its appearance in the *form* of a mathematical proof--the ritualistic aspects of proof--rather than the *correctness* of the argument. They presented students with a false argument to a given mathematical statement and then examined the students' evaluations of that argument. They found that "many students who correctly accepted a general-proof verification did not reject a false-proof verification; they were influenced by the appearance of the argument--the ritualistic aspects of the proof--rather than the correctness of the argument" (p. 49).

The "ritual proof" misconception, however, does not have to manifest itself in such a severe behavior as the judging of mathematical arguments on the basis of their appearance only. For example, on many occasions during the beginning period of a teaching experiment, either in a class discussion or in a personal exchange, students have asked whether a certain justification is considered a proof. When asked to explain the motivation for their question, the students indicated that although they are convinced by the justification, they have doubts whether it counts as a mathematical proof, for "it does not look like a proof." Typically such doubts are raised when the justification is not communicated via mathematical notations and does not include symbolic expressions or computations, even though the argument itself is quite sound by the usual mathematical standards; it is just that the argument does not "look" like a proof.

The Symbolic Proof Scheme. Justifications which use symbols as if they possess a life of their own *without reference to their possible functional or quantitative relations to the situation* characterize the symbolic proof scheme. The power of symbols is well known, but when symbols are empty of meaning, or bear no relationship to the situation for which the symbols were introduced, their use can be counterproductive. For example, it is not uncommon for linear algebra students to interpret the inverse of matrix A as the fraction 1/A, and attempt to reason about the inverse matrix as though it were a fraction.

Perhaps the most devastating consequence of the symbolic scheme is the common behavior of approaching problems without first comprehending the problem situation and its task. It is not unusual to find that immediately after reading the problem, many students begin their solution with some sorts of symbol manipulation of any expressions involved, with little or no time spent on comprehending the problem statement. Students' actions take place quite haphazardly without a clear purpose and without the formation of a coherent image of the problem situation. So, for example, many attempt a solution without knowing the meaning of some of the terms used in the problem statement, and many others are unable to articulate the exact task they were to accomplish. For these university students, the symbol manipulation rules they acquired in their earlier school years define the essence of their mathematical world; quantitative comprehension and sense making, wherein lie the value in representations by symbols, were absent from this world.

The Empirical Proof Schemes

These proof schemes are based solely on examples. As with the authoritarian proof scheme, reasoning based on examples is not entirely bad. Mathematicians value examples highly (see, e.g., Halmos, 1985). Psychologists nowadays note that natural concept formation is based on examples, and sometimes on rather special examples (Medin, 1989). But as Sfard points out, mathematics students must become "sufficiently mature in the mathematical culture" to appreciate the role of definitions in mathematics (1992, p. 47). A similar maturity in the mathematical culture should lead to an awareness of the tentative nature of results suggested by examples.

Inductive Proof Scheme. When students ascertain themselves and persuade others about the truth of a conjecture by evaluating their conjecture in one or more specific cases, they are said to possess an inductive proof scheme. Every teacher has likely observed the dominance of this proof scheme among students, and research corroborates this observation. For example, Chazan (1993) has observed the existence of the inductive proof scheme among U.S. high school students. Martin and Harel (1989) found that more than 80% of their preservice elementary teachers considered inductive arguments to be mathematical proofs. Even with mathematics majors, who presumably are more sophisticated than the high school students or the preservice elementary teachers, the inductive proof scheme is common.

The Perceptual Proof Scheme. This proof scheme fits, for example, many geometric justifications that might be given by younger students. The perceptual proof scheme is based solely on visual or tactile perceptions. For example, a student may examine an isosceles triangle and decide that the base angles are congruent just by visual examination. Older students might be convinced that the medians of a triangle are concurrent by looking at several computer-generated examples, and they might attempt to convince others by showing them similar examples.

The Theoretical Proof Schemes

The Transformational Proof Schemes. The general characterization of these schemes is that students' justifications attend to the generality aspects of a conjecture and involve mental operations that are goal oriented and intended-anticipatory. They are the foundation for all theoretical proof schemes. Here is an example of transformational reasoning from a case study of a fourth-grader (by GH):

> I asked Ed to think of a triangle with two equal angles and describe what he thought the relationship between the sides opposite them. Ed responded almost instantly that the two sides must be equal. I asked Ed to explain to me how he had arrived at this conclusion. Using his hands to describe the triangle, Ed said something to the effect that if one angle (he puts one forearm horizontally and moves the second forearm diagonally to it) is equal to the other angle (switches between the forearms' positions), then the two sides (he puts the two forearms diagonally to form a triangle) are equal. When I continued to press Ed for more explanation, he went on to say: If you launch a rocket from this side (pointing to his right elbow and moving his right forearm diagonally to indicate the direction of the rocket) and at the same time you launch another rocket from this side (pointing to his left elbow and moving his left forearm diagonally to indicate the direction of the other

> rocket), the two rockets will collide and explode at the vertex of the triangle. Their parts will go down exactly in the middle of the triangle and make two little triangles. When you put these triangles together, one on top of the other (he lines up his two hands along the two little fingers and then opened and closed them several times), these two sides would be equal.

Notice the generality of the thinking and its basis in mental operations. Note also that the thinking could easily be turned into the common mathematical proof (since Ed was a fourth-grader, he was not asked to do this).

The transformational proof schemes classification includes three types of transformational proof schemes. Ed's justification illustrates a spatial-images proof scheme, which in general is characterized as a transformational proof scheme in which the context of the justification is of images from spatial intuition.

"Symbolic-transformational proof scheme" is our current label for an encapsulated transformational proof scheme that has become a heuristic in devising mathematical justifications. Repeated applications of transformational proof schemes, if reflected upon, can potentially result in the formation of proof heuristics. Hence, a symbolic-transformational proof scheme is a proof heuristic abstracted from the experience of applying transformational proof schemes. Here is an example, in which an older student transforms the given algebraic expressions into mental images related to graphs:

> Prove that for $x \geq 0$, $\log(x+1) \leq x$. He first converted this inequality into its equivalent $x+1 \leq e^x$, then he said: "Both functions [$x+1$ and e^x] are increasing but e^x goes faster. At zero they are equal, so e^x must be greater."

This student then translated this thinking into a more standard mathematical proof form.

One particularly important example of the symbolic-transformational proof scheme is this: To prove or refute a certain conjecture, the conjecture is represented algebraically and symbol manipulations on the resulting expressions are performed, with the intent to derive relevant information that deepens one's understanding of the conjecture and potentially leads to its proof or refutation. In this activity, the individual does not necessarily form conceptual images for some or all of the algebraic expressions and relations that result in the process. It is only at critical stages in this process--viewed as such by the individual--that the person intends to form such images.

The third transformational scheme is the constructional proof scheme. In the constructional proof scheme a students' doubts are removed by actual construction of objects, as opposed to mere justifications of the existence of the objects. For example, in justifying that the inverse of a square matrix is unique (when it exists), some linear algebra students have preferred a justification in which the inverse of a matrix is constructed, step-by-step, to the usual assume-there-are-two-and-show-they're-equal proof, even though the proof by construction was based on a 2x2 case with numerical entries. The students, most of whom realized the drawbacks of arguments based on specific numerical cases, regarded the argument with the specific case as a generic argument and preferred it because "you can see how it works."

The Structural Proof Schemes. The general characterization of these schemes is that they are special transformational proof schemes in which conjectures and facts are representations of situations from different realities that share a common structure. The structure is characterized by a collection of accepted facts. There are three subcategories, which will be described only briefly here. It is important to keep in mind that these must be transformational in nature; otherwise there is the danger of resorting to rote memory in settings where they could be used. The postulational proof scheme is a structural proof scheme in which the structure is characterized by a collection of permanently accepted facts. This scheme is essential in studying the theory of vector spaces, for example. The spatial-postulational proof scheme is a postulational proof scheme whose realities are based in intuitions of space. The postulates in Hilbert's *Grundlagen der Geometrie*, for example, could provide the characterization with which to justify statements in geometry. Finally, the axiomatization proof scheme is a structural proof scheme in which the structure is characterized by a collection of tentatively accepted facts. This scheme is essential in studying questions of consistency, independence, completeness, etc.

Implications

The symbolic and ritual proof schemes, grounded as they are in meaningless symbol manipulation or surface features, have nothing to recommend them; perhaps with a greater emphasis on the giving of justifications instruction can help students to avoid them. Students must be educated to value and to want to know justifications; the source of the results, not just the results, must be emphasized. The authoritarian proof scheme, on the other, hand is a two-edged sword. In the culture of schools or of knowledge acquisition, it can be valuable. The concern is to move away from a complete reliance on it and its suffocating effect on the giving of justifications. For example, in the teaching experiments, a conjecture was no longer labelled "theorem," simply because the label "theorem" seemed to reduce the students' effort, willingness, and even the ability for some students to justify the conjecture. The label "theorem" apparently rendered the relationship into something to obey rather than to reason about. The use of small groups, in which there is no obvious authority figure, seems to foster more openness to evaluating justifications; there the student is a more genuine partner in justifying statements than in a teacher-led justification. The empirical proof schemes, with their roots in everyday thinking, are important and valuable. The inductive proof scheme is so strong, however, that instruction must deliberately combat it to show its defects.

To become "sufficiently mature in the mathematical culture" or to progress toward Poincaré's what-one-wants-the student's-mind-to-become, it is clear that a student must move beyond the external and empirical proof schemes. Of greatest importance is instruction that promotes transformational proof schemes, since these are the foundations for the theoretical proof schemes. The teaching experiments suggest that much progress can be made by designing instruction on carefully chosen problems and making justifications an accepted part of the class routine.

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Seeing, Doing and Expressing: An Evaluation of Task Sequences for Supporting Algebraic Thinking

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Abstract

In this paper we describe a research study in which we set out to explore students' use of visual strategies, the circumstances under which links are made between symbolisation and visualisation and the influence of computer use on these strategies and linkages. In this study we have been investigating student approaches to a sequence of algebraic problems presented with visual information. Our comparative analysis of students' responses to three different task sequences involved documenting the trajectory of visual and symbolic approaches, attempting to identify the form in which they occurred,why they occurred and if they were inter-connected. The sequence of activities included work on the computer and we explored if and how interactions with the software connected with other approaches. To illustrate our methodology and findings, we present data from three students who worked through the problems in different mathematical settings.

Background

It is generally reported that students of mathematics, unlike mathematicians, rarely exploit the considerable potential of visual approaches to support meaningful learning (see, for example, Bishop 1989; Dreyfus 1991). Where the mathematical agenda is identified with symbolic representation, students are reluctant to engage with visual modes of reasoning. Conversely, when powerful visual images are present, students tend to exhibit a preference for solving problems simply by perception without mobilising any mathematical knowledge (Hillel, Kieran and Gurtner, 1989). Students' reasoning tends to be compartmentalised: they operate in one or other mode without making links between the two (Presmeg, 1986; Hoyles and Noss 1989).

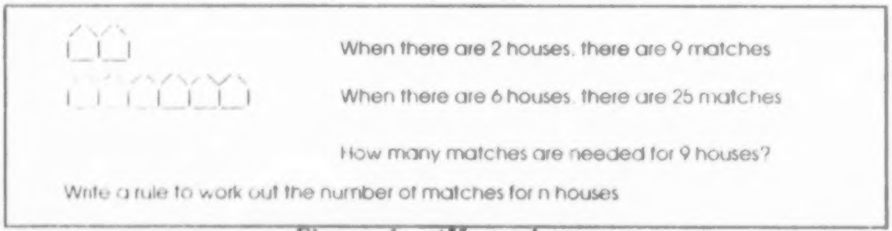
In many ways, these findings are unsurprising. Mathematicians know what to look for in a diagram, know what can be generalised from a particular figure and so are able to employ a particular case or geometrical image to stand for a more general observation. Our question is, how can students best be encouraged to share in these ways of thinking — what systems of support can we offer which will encourage them to make connections between visual and symbolic representations of the same mathematical notions. Underlying this question is a fundamental assumption that permeates our research, that mathematics learning involves students in constructing connections, in linking new mathematical knowledge with what they already know — both about the system of mathematics itself and with knowledge derived from other domains. We have particularly focused on mathematics learning in computational settings, settings which open up new possibilities for incorporating visualisation into the practice of mathematics: Computers offer the potential to operate on images with the kind of rigour which has hitherto been reserved for the symbolic; visual images can be externalised[1] and rendered manipulable. Previous experience tells us that we cannot assume that this potential will necessarily transform students' mathematical thinking — at any rate, it is unlikely such a change will be realised spontaneously. We therefore set out to search for circumstances under which students come to move more freely between visual and symbolic representations and to investigate the role the computer might play in this process.

The study

Our investigations, carried out during the project *Visualisation, Computers and Learning*,[2] focused on students working through carefully-sequenced activities designed to exploit the visual alongside the symbolic in pursuit of a range of previously specified mathematical goals. Specifically our aims were:

- to map students' visualisation strategies in two mathematical domains: geometry and algebra;
- to identify if and how links are made between symbolisation and visualisation;
- to identify if and how strategies and links between them are influenced by computer use.

In this paper we describe the algebra strand of the project in which the mathematical focus was the study of *Number Patterns*, a topic commonly used within the UK as a vehicle for introducing students to algebraic notation and functional invariance. An example is presented in Figure 1.



When there are 2 houses, there are 9 matches

When there are 6 houses, there are 25 matches

How many matches are needed for 9 houses?

Write a rule to work out the number of matches for n houses

Figure 1: 'Houses' sequence

The idea behind these activities is that students will identify relationships within numerical patterns derived from spatial situations, perhaps express these in natural language but ultimately formulate a symbolic generalisation. Through such a process of doing, seeing and expressing, it is argued that they will build algebraic meanings for the symbolic notation. The UK National Curriculum suggests a sequence of progression whereby students work from simple one-operation linear functions to quadratic functions. Within the curriculum guidelines, as students become more proficient they are expected to move from paper and pencil on to spreadsheet explorations of number patterns. We wanted to compare this computer-added approach with one in which computer use was integrated throughout a task sequence. Additionally, since spreadsheets offer the possibility for symbolic interaction but little chance for manipulation of visual objects, we wrote a Logo microworld, Mathsticks, in which students would have the opportunity to generate computer-mediated visual and symbolic representations[3]. Thus, three task sequences with the same mathematical content were designed.

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[1] We accept that in this process the images will change but nevertheless suggest that these externalised images are worthy of investigation

[2] This project was funded by the Economic and Social Research Council [Grant Number R000234168].

Task sequences: The first task sequence was based on existing materials found in the school in which we conducted the research. This computer-added task sequence (the CAT) involved paper and pencil work followed by activities using a spreadsheet. In two computer-integrated task sequences, the spreadsheet CIT and the Mathsticks CIT, computer use was incorporated at all levels. All the sequences followed a common pattern – individual semi-structured interview, pair work, group work, pair work, individual semi-structured interview. Both the individual and pair work were based around a common set of tasks where the students worked on identifying and expressing general patterns underlying different number sequences presented through both visual and numeric data[4] as illustrated in Figure 1. Thus the aim for each task was that students constructed and justified a general method to calculate values for the nth term. In the group work, two pairs came together to discuss their previous activities and to justify any relationships they had identified.

Data collection: Three groups of four students, aged 12-13 years, were selected for case study. The groups, chosen in conjunction with the mathematics teacher, were organised so that each group comprised a similar spread of ability. Each task sequence spread over about six weeks and took up about 10 hours of student time. The data comprised student responses (paper and pencil or computer work together with video and/or audio-recorded discussion) in each of the five settings of the task sequence. The individual interviews were task-based where students' written responses were followed up and probed by the researcher. During pair and group work a researcher was present as a participant observer with the role of teasing out students' intentions, strategies and explanations. At no time did she give direct assistance in relation to the mathematics although she did provide syntax advice when this was specifically requested. The data were synthesised into detailed case histories describing the trajectory of each student working through a task sequence. These case histories were then interrogated to find out *if* students' goals, strategies and outcomes shifted and, if so *how* and *when*.

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[3] Mathsticks was designed in conjunction with Richard Noss. We do not have space to descibe in detail here but intend to demonstrate its main features during the presentation (see also Noss, Healy and Hoyles in press).

[4] Pilot interviews suggested that presenting terms not in sequence resulted in greater attention to the visual data.

Analysis of the case studies

Our case study students were relative novices in this area: Our aim was to examine how all the factors in the learning setting interacted to facilitate (or to inhibit) students in evolving a coherent knowledge system from an initial set of disconnected fragments of mathematical ideas. We began by classifying the conceptions and strategies that students applied to these kinds of problems from a review of the relevant research literature (e.g. Stacey, 1989; MacGregor and Stacey, 1992; Orton and Orton, 1994) and from extensive pilot interviews we conducted before undertaking the case studies. The research studies we surveyed tended to report students' difficulties and errors as evidenced in "one-off" situations. Two particular strategies were extensively documented: First, a tendency to make false assumptions of direct proportionality between terms when working on linear sequences of the form $f(n) = an + b$ $(b \neq 0)$, and second an overemphasis on recurrence relationships in one variable. The problem with focusing on student errors in this way is that the emphasis inevitably is on what students cannot do rather than where these strategies would have worked (e.g. when $b=0$ in above example) and how students could move on to more generally applicable methods.

Our agenda was to go beyond analysis of student behaviour and investigate their mathematical thinking-in-change. The framework we eventually devised incorporated the strategies previously documented but recast as a set of what we termed *construction approaches*, each of which was deemed to represent the evocation of a set of cognitive resources through which a student had tried to make sense of the activity. We distinguished iconic and symbolic approaches and within each category identified four different ways in which students organised and manipulated the data as they attempted to construct a generalisation. The approaches are presented in Table 1 as a two by four matrix in order to point up the mathematical equivalencies between the horizontal cells. However, it was clear from our observations that these equivalencies were not necessarily apparent to the students and our aim in the next phase of analysis was to trace the evolution in a student's thinking and to search for the conditions where connections were made between approaches.

Maps of student approaches: To represent how student approaches evolved over the course of a task sequence, a series of maps was constructed from the basis of the analysis framework. One map for each of the five research settings was produced for all the three case-study groups. It showed pictorially all the approaches used – rectangular shapes indicating symbolic approaches, oval iconic ones, and their frequency shown by the thickness of a shape's boundary. Approaches associated with the final generalisation made were shaded. A map also showed the connections made between approaches. Connections took the form of exchanges where students explained the responses associated with one approach by an explicit reference to the resources underpinning another. Thus a connection was a mathematical justification (or a refutation if an incorrect approach was rejected or debugged) – students verified that one construction was consistent with an alternative way of viewing the problem. A connection was represented on a map by a line linking the relevant approaches.

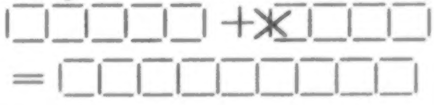
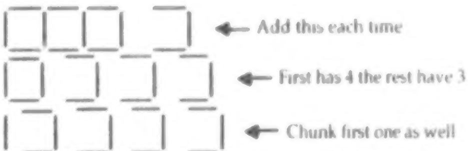
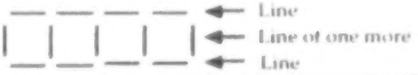
Symbolic	Iconic
Counting	**Eidetic**
Counting the matches in an unstructured way	Focusing on perceptual rather than mathematical properties of the data: *"The star is like a cross from noughts and crosses and a religious cross."*
Operating on terms	**Combining diagrams**
Calculations using a known term or terms to obtain a target term: *1) "there are 16 matches in 5 so there will be 48 in 15, you times by 3"* *2) "to work out 7, I did 10 add 13 because 3 had 10 matches and 4 had 13"*	'Chunking' of known terms to obtain another: 
Operating on differences between terms	**Inter-term**
Calculations based on the numerical difference between consecutive terms: *1) "4 is 13 because you add 3 each time"* *2) "I added 30 because the difference between 5 and 15 is 10 so you add 10 3's"*	'Chunking' based on a relationship between terms:  Add this each time First has 4 the rest have 3 Chunk first one as well
Operating on variables	**Intra-term**
Calculations based on a relationship between dependent and independent variables: *"You times the number of boxes by 3 and add 1"*	'Chunking' based on a relationship within a term: 

Table 1: Classification of Student Approaches

A Snapshot of our results

To illustrate our methodology further and to give an indication of the differential influences on student approaches of the different task sequences, we present the data of the approaches of three students, one from each of the case study groups: Jodie, who worked on the CAT task sequence: Lesley, a member of the spreadsheets CIT; and Tombana, one of the Mathsticks CIT students. First we present the maps derived from the first and last interviews with each of these three students (Figures 2 - 4) to illustrate any changes made in each individual's configuration of approaches.

Both sets of interviews were concerned with two-operation linear sequences. In the first interview, one task was given, while two tasks were presented in the final interview. The maps show that, in their first interviews, all three students used a number of different approaches and made no connections between them. Jodie and

Tombana both failed at any point to exploit the iconic data. They also constructed a final generalisation that did not lend itself naturally to algebraic expression — in fact, the approaches they chose were those mentioned in the literature as being frequently associated with errors[5]. Lesley, on the other hand, showed an initial preference for iconic approaches, and her final generalisation involved an intra-term relationship.

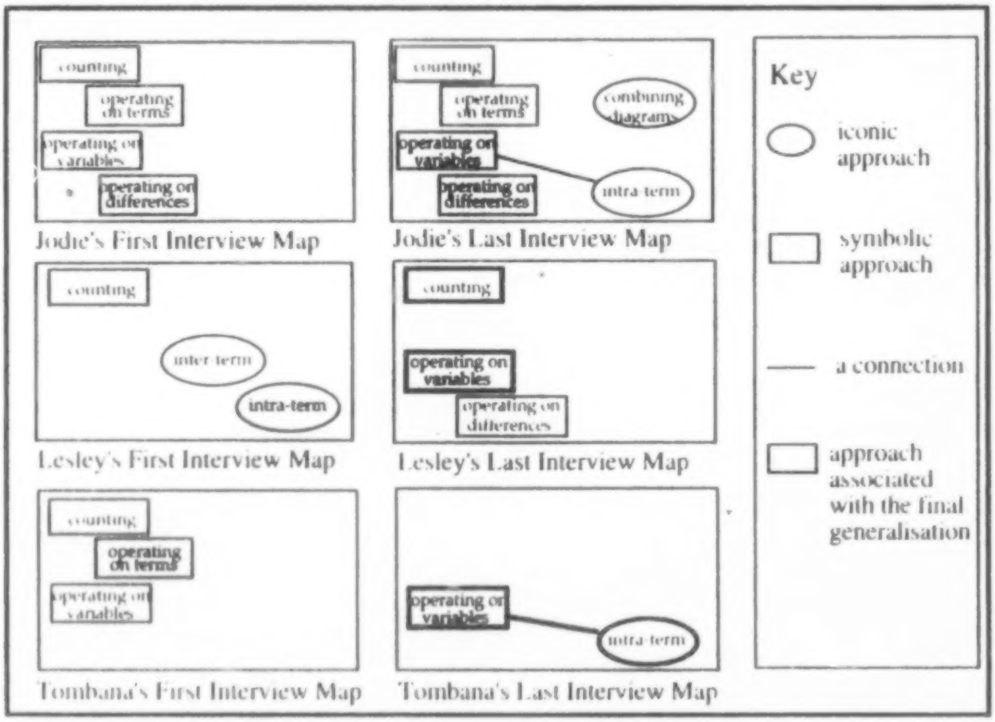

counting
operating on terms
operating on variables
operating on differences
Jodie's First Interview Map
counting
operating on terms
combining diagrams
operating on variables
operating on differences
intra-term
Jodie's Last Interview Map
counting
inter-term
intra-term
Lesley's First Interview Map
counting
operating on variables
operating on differences
Lesley's Last Interview Map
counting
operating on terms
operating on variables
Tombana's First Interview Map
operating on variables
intra-term
Tombana's Last Interview Map
Key
iconic approach
symbolic approach
a connection
approach associated with the final generalisation

Figure 2: First and last interview maps for three case-study students

The maps of the student approaches in the final interview suggest that the students responded rather differently to the same type of task at the end of the sequence. In contrast to her first interview, Jodie adopted both iconic and symbolic approaches, but only occasionally constructed connections between them. Her final generalisations were associated with two different approaches, suggesting she had not yet developed a consistent pattern of working. The two CITs students, on the other hand did develop consistent sets of approaches and their final generalisations both involved operating on variables. However, while Lesley actually made more use of iconic data at the beginning of the task sequence than at the end, Tombana's final, more connected map represents how she used the iconic data as a means both to construct and to justify the mathematical relationships conveyed by her algebraic symbolism. For all three students, there is some indication of a move away from approaches associated with arithmetic methods and a reliance on specific cases towards the production of general functional relationships, although only Tombana's final map shows the set of approaches that the activities were intended to engender. However, it is clear that consistency amongst the final approaches, the use of visual or symbolic reasoning and the construction of justifications by connecting approaches all differed substantially across the three task sequence.

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[5] Note that, at this level of analysis, it is not possible to ascertain whether a student's final generalisation was actually correct. The more complete versions of the maps (which we do not have space to reproduce here) contain this information, although this was not our main concern. All approaches, potentially at least, can lead to both right or wrong answers. Our focus was how the students were thinking about the tasks in hand.

Of course, the first and last maps show only snapshots of the entire story. Making sense of the differences between the two involves tracing a student's path through the other settings and considering her interaction with other students (who mobilise different resources) and with the media of the setting. We do not have space here to present the maps for all the intervening sessions, instead we consider briefly these three students' interactions during the pair settings and the different influences of tools available within each task sequence. During the pair setting in which Jodie had access to paper and pencil only, her manner of working was eclectic — she simply chose whatever approach most quickly seemed to lead to some (any) generalisation. For Jodie, the spreadsheet task in the CAT sequence added little, in fact she felt this activity was about learning to use the computer and not connected to the previous work at all. In contrast, the spreadsheet CIT sequence seemed to have encouraged Lesley to focus on symbolic aspects. In her pair work a consistent style quickly emerged characterised by "pattern-spotting" where relationships in tables of numeric data were identified without apparent appreciation of the need to connect these to the structures underpinning them. It was not that Lesley was unable to use the iconic data, but that the goal of the task became transformed to become the construction of spreadsheet rules with the result that the iconic information had little relevance. What is significant is that this pattern of working persisted into the final paper and pencil based interview where Lesley's original use of approaches was no longer apparent.

We conjecture that the problem in both these task sequences(the CAT and the spreadsheet CIT) is that there is a gap between seeing a pattern and the means of expressing the pattern which students frequently chose not to cross, with the result that their thinking remained compartmentalised. The Logo tools opened up a new set of possibilities. In Mathsticks the means of expressing actions is firmly soldered to the activity: students can interact with virtual matches and, as they do so, a symbolic trace is produced, or if, they communicate in symbolic terms, a corresponding visual trace is generated. In Tombana's interactions with Mathsticks those tools were bought to life as she, along with her partner, constructed relationships by first systematising her actions to produce a visual display of the pattern, identifying REPEAT structures in the symbolic representation automatically produced by her actions and finally using these as a basis for building general Logo procedures. She was also able to work from her symbolic representation back to the visual — recognising, for example, that the addition of an extra match within a REPEAT loop could change a sequence of boxes to a sequence of houses. Thus in Mathsticks, seeing, doing and expressing become inextricably linked: since a student's visualisations are coupled with the symbolic; mathematically speaking they are one and the same. The resulting cognitive residue for Tombana (along, in fact, with all the other members of the Mathsticks case-study group) was a robust stance to number pattern problems which spread beyond the boundaries of the computer setting.

Final remarks

Our results suggest that students bring numerous resources to mathematical situations and that the approaches they choose to apply to any given problem vary according to setting. It seems foolhardy therefore to deduce that a student is incapable of using a particular approach on the basis of observations from just one setting. On the contrary, we have found that the adoption of a particular set of approaches depends, among other things, on the tools available. The Mathsticks microworld seemed most likely help students appropriate our intended learning aims, to provoke them to shift from a pragmatic to a theoretical stance to the number patterns (Balacheff 1986). Our contention is that the route towards construction of mathematical meanings — in this case algebraic meaning — is best supported by tools designed to help bridge the gap between action and expression; to scaffold movement *to and fro* between the visual and symbolic in much the same way as spontaneously achieved by the mathematical cognoscenti.

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THE ROLE OF PRIOR CONCEPTIONS IN TEACHERS' RESPONSES TO STAFF DEVELOPMENT: A SYNOPSIS OF CASE STUDIES OF THREE MIDDLE SCHOOL MATHEMATICS TEACHERS

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The purpose of the study was to document the process of teacher change within the context of an inservice. My focus was the individual teacher's perspective of change, including beliefs and conceptions of what it means to teach and learn mathematics. Therefore, I accompanied three teacher-participants through a six-week summer institute and several seminars during the following academic year, collecting data prior to, during, and after the inservice experience. Through preparation of case studies I identified themes that emerged across cases and concluded that prior conceptions held by staff developers as well as by the teacher-participants played a definitive role in the teachers' adoption of the innovative instructional method modeled in the inservice.

In response to the call for reform in mathematics instruction in the United States, standards for revising curriculum and evaluation (National Council of Teachers of Mathematics [NCTM], 1989) and for teaching mathematics (NCTM, 1991) have been promulgated among the nation's teachers from elementary through secondary levels [grades 1 through 12]. Unlike the mathematics reform movement of the 1960s, this one does not rely on "teacher-proof" curriculum materials; instead, the teacher is seen as key to reform. As Fullan and Steigelbauer (1991) commented succinctly, "Educational change depends on what teachers do and think--it's as simple and as complex as that" (p. 117). In the effort to change teachers' thinking about mathematics instruction, the primary outreach to practicing teachers is the inservice, to which they bring their prior conceptions of effective mathematics instruction. A question that arises, then, is: How do teachers' prior conceptions interface with inservice education?

Method

The research reported here examined this question through case studies of middle school mathematics teachers [teachers of grades 6 - 8] who were participants in Project Discovery, a statewide initiative sponsored by the National Science Foundation and by the State of Ohio. An intensive, long-term mathematics inservice, it included a six-week summer institute on the inquiry method of instruction, and several follow-up seminars. Since the purpose of the study was to document the process of teacher change within the context of inservice education, I conducted extensive interviews with and classroom observations of three participants prior to and during the summer institute, actually attended the institute myself as a full-time participant, and then made several two-day visits to their schools during the following academic year. From this methodological stance of participant observer, I documented through case studies the interaction of the individual teacher with the staff development experience, in order to give voice to the teacher-participants and to better understand the complexities of teacher change.

The subjects, termed "teacher-participants," were:

•Beverly, a Caucasian female, secondary-certified, with 17 years of teaching experience in a small rural town;

•Imani, an African-American female, elementary-certified, with 4 years of teaching experience in urban city schools;

•Scott, a Caucasian male, elementary-certified, with 15 years of teaching experience in a small town that serves as a residential adjunct to a large city.

As can be noted, subject selection was purposeful and heterogeneous, covering a range of teacher variables.

In the research setting, an inservice offered for teachers of middle school mathematics, participants were immersed in collaborative problem solving: inquiry problems that modeled open-ended, hands-on problems with multiple solutions and extensions. The university instructors expected the teachers to construct their own sense of inquiry teaching through their experiences with the problem solving, through class consensus, and through creating lesson plans that incorporated an inquiry approach.

Theoretical Framework

Constructivism, a "theory of active knowing," holds that knowledge is constructed as the engaged thinker attempts to organize his/her individual experiential world (von Glaserfeld, 1988, p. 33). What is particularly relevant in this theory to the role of prior conceptions in teacher change is its claim that new information is not passively received but, rather, reviewed in relation to an already organized operating system. If the new information is not seen to fit into an already accepted category and thus cannot be assimilated into the system-as-is, a disequilibrium or perturbation occurs. Within the context of this study, the inservice experience was considered a potential cause of disequilibrium, one that could stimulate change in the form of either assimilation or accommodation.

Moreover, the Project Discovery Mathematics Summer Institute placed the participants in a learning environment shaped by constructivist theory. Instead of offering a set of lectures on the inquiry method of instruction or even problem solving strategies, the instructors immersed the teachers in problem situations which required them to directly encounter and explore the mathematcs embedded in the situation. This was their introduction to inquiry teaching. The participants were expected, within a small group setting, to collaboratively make sense of the given problem situation, test solutions given by the class, and construct new mathematical understandings. This view of teaching and learning differs markedly from that held by most mathematics teachers (Romberg, 1986), which proved unsettling as the participants considered how, or even whether, to implement the instructional philosophy being modeled.

Finally, constuctivism framed the data collection as well as the method of analysis. The data collection acknowledged the teacher-participants' prior knowledge, included information on how they engaged in making sense of innovative instructional methods, and allowed for the expression of personally-construction meanings. The use of the case study as a method of analysis honors the view that response to a learning environment is necessarily individual and unique.

The Role of Prior Conceptions

Incongruent Mathematical Philosophies

Staff developers as well as participants bring to the inservice setting philosophies of mathematics that drive their theories of instruction and learning. In Project Discovery the instructors operated from a holistic view of mathematics--a terrain without fixed borderlines between topics, terrain to be explored rather than curriculum to be covered--with "doing mathematics" defined as delving into a rich mathematical situation and drawing from several areas of mathematics to resolve it. For the teacher-participants "doing mathematics" generally referred to applying fixed procedures, sequentially and linearly, within well-recognized borders. What was experienced by the teacher-participants was a lack of connection between instructional philosophies, between what was modeled in the inservice and what was expected of them in "real world" classrooms. Imani commented:

> I teach my kids math is like a chain. You just keep linking the chain up. Everything you get, you just link to the chain. But if I'm not telling where, or giving them good examples so they can discover where they can link it to . . .

The teacher-participants expressed a dissonance between their conceptions of teaching/learning mathematics and those of the instructors, a mismatch of purposes and objectives. The teachers looked for objectives that were compatible with the tradition of school mathematics (Cobb, Wood, Yaekel, and McNeal, 1992), with its officially-mandated curriculum, its assessment of discrete items, and its clearly defined borders. Significantly, their view of school mathematics holds for the general population of teachers (Brown, Cooney, & Jones, 1990; Romberg, 1986). It is likely, therefore, that the incongruity that emerged between the staff developers' mathematical conceptions and those of the participants is a fundamental feature of inservice education.

Predisposition Toward Inservice Education

The dissonance was exacerbated by their prior conceptions of inservice education. Within a staff development setting, they expected to "pick up things": to select from the array of new techniques those that suited their classrooms and to "insert" them into their existing practices. Beverly commented, "I'll pick up things and I'll try, and I'll throw away and I'll keep." As a consequence, the common mindset of the teacher-participants was to analyze the innovative instructional method, to see it not as a whole philosophy but as a sum of distinct parts to be considered separately and adopted separately. What they expected to acquire was not a different approach to teaching but rather discrete lesson techniques that could enhance and expand their existing practices.

Personal Definitions of Teaching Mathematics

A teacher's established core of practices, and the conceptions that meld it, represent years of work, of learning and adjustment, for "practitioners' own sense of self is deeply embedded in their teaching" (Rudduck, 1988, p. 208). As seen in this study, teachers' prior conceptions mediated the inservice material, determining its "fit" and its function in relation to exisitng practice. In the process of determining whether or not to implement the inquiry method, they actively looked for alignment with their personal conceptions of good teaching and, if not found, they either modified the innovation or rejected it. Observations and interviews showed that any change in classroom instruction corresponded to the individual's definition of teaching and what it means to teach mathematics in the school setting.

An example was Scott's conviction, expressed before attending Project Discovery, that only mathematics that could be applied to real problems was worth teaching: "If I tell them about the Pythagorean Theorem without telling them where they'll use it, why learn it? It's just mental gymnastics at that point." Later, when he was, indeed, engaging his class in problems that related to number patterns, for example, problems that had no direct relevance to applications, he explained that such problems taught his students "to think," which had the most direct application he knew to "real life" problems. He had achieved alignment with his personal philosophy by seeing his earlier goal subsumed into the larger goal of "teaching them to think."

Primacy of Officially-Mandated Curriculum

For all practical purposes, the official curriculum as expressed through school guidelines defined school mathematics, and the primary responsibility of teaching, as perceived by each teacher-participant, was to cover that curriculum. Inevitably linked to curriculum was preparing students for official district and state assessment. Such assessment constituted accountability for the teachers as well as for their students.

How to address the very real issues of curriculum and assessment proved to be concerns, if not outright frustrations, for the teacher-participants. Imani wanted to know "how to relate this [inquiry approach] with the book that I have, because in the real world we have a timeline and things that have to be done," and "here [at the Summer Institute] they're not on a timeline, but when I jump back into reality at my school I'm on a timeline, and it's not that I'm pushing toward the test, but I have to give kids those tests." The common perception was that the staff developers failed to take into account the working situation of the participants.

With regard to the long-range effect of the inservice, it is notable that the official curriculum, both content and instruction, remained unaffected by whether or not the teacher-participant implemented the inquiry approach. That portion of mathematics which corresponded to official guidelines and which was to be assessed eventually by official examination, the "real" mathematics, maintained the format and style adopted by the teacher before the staff development intervention. In the case of Scott, who came to consider himself a wholehearted proponent of the inquiry philosophy, after the inservice experience two ongoing but separate curriculums were seen to be operating in his classroom. He allotted the majority of his lesson time to inquiry problem solving and those strategies advocated during the inservice: cooperative groups, teacher as facilitator, encouraging multiple answers and having students present their solutions to the class. But he maintained as well a textbook-driven portion of his lesson time, with problems and pace dictated by the textbook, teacher-centered explanations, with topics treated discretely and fragmented into individual skills. Leaving this textbook portion intact assured him that the official curriculum was covered, which remained his primary responsibility in his definition of mathematics teacher.

The Classroom as Validator

When I asked Beverly at the end of the Summer Institute what changes, if any, had occurred in her thinking as a result of the inservice experience, she responded emphatically:

> That question won't be answered until I actually get back into teaching again. I just feel like I want to try some things and see if they work for me, see if they work for my kids. . . . So I don't see myself answering this a lot, not until I've started school. That will tell me.

This view of classroom reality as the crucible where new ideas would be tested and validated or rejected was shared by the other teacher-participants. To determine if and how inquiry would fit into established practice, they felt they had to experience it first in their classrooms and see if it "works for my kids." Riseborough speaks of the "often underestimated symbiotic relationship between teacher and pupil" and points out that teachers "learn from pupils, they learn what is possible and what is not" (quoted in Ball & Goodson, 1985, p. 17). Certainly, the feasibility of inquiry teaching was affected by such structural constraints as the school schedule, availability of materials, and preparation time. But the teacher-participants saw student response as a more significant factor--both in shaping the implementation process and, ultimately, in determining the viability of the innovative instructional method.

Implications

It is a defining characteristic of inservice teachers that they bring to the staff development setting an established practice, including the conceptions that underlie that practice. From their vantage point of direct contact with classroom reality, the efficient operational mode is practical and classroom-oriented: select those discrete units that are congruent with prior conceptions and insert into existing practice. Furthermore, conceptions mediate new material as teachers strive to maintain intact their personal interpretations of mathematics teaching and to fulfill their perceived primary responsibility of covering offical curriculum. The teachers in this study felt professionally responsible for "reaching" as many students as possible, engaging them during the class period, and preparing them for external testing--hence, the teachers' vulnerability to student response and their adherence to official curriculum. How an innovative method maps onto district guidelines, therefore, and how it corresponds to student expectations relate directly to implementation of educational reforms.

Given the power and persistence of prior conceptions (Duffy & Roehler, 1986; Wallace & Louden, 1992), those involved in mathematics reform need to address the various sources of dissonance between established teacher practices and innovative methods, especially those created by incongruent mathematics philosophies. Otherwise, those conceptions that underlie existing practice can lead to a re-shaping of an innovative program into a form that aligns more comfortably with the status quo and can inadvertently sabotage teacher change.

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The use of levels of subordination to help students gain fluency in mathematics

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Abstract: *There is a current debate concerning the desire for children to have regular practice in mathematics lessons and to gain fluency over areas of mathematics such as numerically and algebraic manipulation. In this paper, I develop a theoretical model of how fluency can be achieved through the notion of 'subordination' and the role this plays in successful learning outside the classroom. I discuss ways in which this notion can be brought into the classroom, and how successive levels of subordination can help a learner become so fluent in a skill that they require little or no conscious attention when employing it - a process I name 'functionalisation'.*

The issue of gaining fluency in areas of mathematics has been a subject of debate recently in the UK[1] (Barnard and Saunders, 1994; Ernest, 1995; LMS et al, 1995). This has often become a debate between 'progressive' and 'traditional' teaching methods, as if traditional methods are the only way of achieving fluency in mathematics. Although there are exceptions, 'traditional' methods can lead to mechanistic repetition with little understanding, and alienate many people from mathematics. 'Progressive' methods, such as the use of investigations can involve children in doing mathematics but sometimes only involves them in practising low-level mathematics. Again, although I am aware of notable exceptions, many investigations, as part of examination coursework, appear to have an expectation of a certain procedure being followed - collect numerical data from particular cases, put them in a table, find patterns in the numbers and express these in algebraic notation. Since an algebraic rule is seen as an endpoint, there is little practice of manipulating and working with algebraic expressions. Also, the potential breadth of mathematical properties which might be noticed and skills practised are sometimes lost because of an almost mechanistic procedure of 'doing investigations'. As a consequence, few skills are practised except for those relating to spotting number patterns (Hewitt, 1992).

Outside the classroom

The driving of a car involves skills which are practised on a regular basis. The nature of that practice involves far more than mere repetition. For example, the movements involved in changing gear - those of hand and feet - are rarely practised by a learner driver when the car is stationary. The aim is not to be able to change gear, but to be able to drive a car in traffic. Changing gear is a necessary in order to achieve this aim. Even when changing gear, a driver has their attention mainly on the road rather than their feet. In fact, attention will drift from one to the other at times, but what is significant is that attention does need to be on the *consequences* of the foot movements, and is not solely on those movements themselves. Thus, this type of practice involves a skill to be learned (feet and hand movements to change gear) being *subordinated* to a different task (the car's movement in traffic). I have developed elsewhere (Hewitt, in press) this notion of *practice through progress* where a skill to be learned is practised whilst being subordinated to progress within some other task. I say that a skill, A, is subordinate to a task, B, if the situation has the following features: (a) I *require* A in order to do B. (This may be an existing necessity or can be created through the 'rules' of a task); (b) I can *see the consequences of my actions* of A on B, *at the same time as making those actions*; (c) I do not need to be knowledgeable about, or be able to do, A in order to understand the task, B.

There are many examples of skills being learned through their subordination to other tasks. Janet Ainley (1995) said *I am reminded... of discussions with teachers who feel that children would need to learn keyboard skills before they could use Logo or a word processor, so that they don't become frustrated by their slow typing. I point out that I developed my (quite considerable) typing ability mainly through programming and writing at the keyboard* (p16). Dewey (1933) talked of the practice involved in developing the human senses: *Sense perception does not occur for its own sake or for purposes of training, but because it is an indispensable factor* *of success in doing what one is trying to do* (p249). He goes on to say that *Training by isolated exercises leaves no deposit, leads nowhere;* (p250).

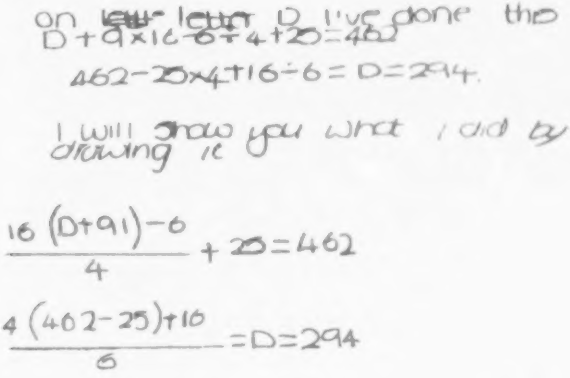
Mathematics classrooms

Laurinda Brown (1991) talks about a series of lessons concerning matrices, where children are learning and practising skills whilst working on other tasks: *If an individual had a problem with plotting coordinates in the early stages of the investigation, it was soon sorted out because of the frequency of use of the data* (p13). I have developed approaches to the learning of formal algebraic notation (OU, 1991; Hewitt, 1994; Hewitt, in press), where the notation is introduced and immediately subordinated to the task of finding my number after I said, purely verbally, various operations which had been carried out. For example, "I'm thinking of a number, I add seven, multiply by five, take three and I get fifty-two". The children have already worked out how to reverse the order and use inverse operations and so can find my number. However, I deliberately give such a long list of operations that they cannot remember them without having a visual reminder. It is at this stage that I write something down for the first time, and write out my series of operations within formal notation, such as the one below. The children have no choice but to go through the notation in order to know what operations were done, and so find my number.

$$6\left(\frac{5(x+7)-3}{2}+8\right)-4=200$$

Emma Brown used this approach with a mixed ability class of 13-14 year olds, and found that all of her students became confident in using and interpreting algebraic notation very quickly. One student who had difficulty with the work, Donna, made several 'errors' (see below). However, what had been subordinated to this task of doing things to an unknown - the use of standard formal notation - was correct. This is not an isolated example. I have found with this activity that what is subordinated in the activity - formal notation - is retained by children over a long period of time.

In contrast to this, after a similar period of time, children become a little uncertain about solving equations, which had been a major focus of attention during the activity.



on ~~lew~~ letter D I've done this
D+9×16−6÷4+20=462
462−20×4+16÷6=D=294.

I will show you what I did by drawing it

$$\frac{16(D+91)-6}{4}+20=462$$

$$\frac{4(462-25)+10}{6}=D=294$$

This is quite a different model of learning compared to traditional repetitive exercises because attention is deliberately taken away from what is being practised and placed on a task in which it is subordinated. Thom (1973), in his discussion of the modern mathematics movement, discussed the possibility that it is not always desirable to make everything explicit by making it the focus of attention. He was critical of the assumption that *By making the implicit mechanisms, or techniques, of thought conscious and explicit, one makes these techniques easier* [his italics] (p197). Practice through progress has purpose because there is a need to carry out the practice in order to gain progress in a task. The model is also different to 'discovery' methods which are based on the notion that because someone has found something out for themselves, they are more likely to remember it. The subordination model acknowledges the need for practice and recognises that meeting something once, albeit by discovery, is not likely to be sufficient for something to be retained over a long period of time. Furthermore, there is a significant difference between consciously discovering a particular skill or property, and using that skill or property fluently in novel situations. Practice by progress is always concerned with subordinating and so applying that skill or property to other situations where it is needed. Thus, subordination not only offers practice but also situates a skill within relevant and changing contexts.

Levels of subordination

I invite you to do this exercise before continuing to read: The word *suboardinasion* is misspelt. Write down (don't just think!) how it should be spelt.

When I have asked people to carry out such an exercise, no-one has commented to me that their attention was placed in the required movements of their fingers in order to write the letter *d*. There are complex manoeuvrings of the fingers required in order to carry out the challenge of drawing a *d*. I can no longer recall my own personal experience of learning to draw this letter. However, I can observe young children engaged in the challenge and notice that it is far from a simple task. I can observe that there is great concentration and effort on behalf of a child when learning to write this letter. I can deduce that the same must have been true for me, and for you. Yet here you are, successfully writing this letter with little or no conscious attention being placed in the physical activation of muscles required to write it. This exercise provides the opportunity to become aware that there are skills you subordinate at an unconscious level. Your attention is placed in a challenge at a higher subordinate level (in fact many levels higher) of writing the correct spelling of a word. In fact, we have become so good at writing a single letter, that we are able to do so at any time it is required and need give no conscious attention to doing it (or such a small amount as to be negligible compared to the conscious energy given to the main task). Gattegno (1971) described such things as *functionings*. I have called the process of something becoming a functioning as *functionalisation*. This process is a result of successive levels of subordination. This hierarchy of levels is only a hierarchy of subordination *within particular situations*. There is no absolute hierarchy. For example, a computer graph drawing package might be used with a task of trying different values for a and b in the equation $y=ax+b$ in order to get a straight line which goes through two points on the screen. With the constraint that it is only through typing in an equation that a line is drawn, choosing the values of the coefficients in the equation $y=ax+b$ is subordinate to the drawing of a particular line. Alternatively, a different computer program could be used, such as *Cabri II*, where a line within a co-ordinate system can be 'picked up' and dragged. As this happens, the equation of the line, which is also on the screen, changes accordingly. Thus, a task could be given to students where they have to move a line on the screen so that the equation reads $y=3x-5$. With the constraint that the equation cannot be changed directly, the position and orientation of the line is now subordinate to the changing of the coefficients a and b in the equation $y=ax+b$.

One example of a chain of subordination is expressed by the following questions a young child may engage in whilst learning their first language. Each new task subordinates the skills developed in the previous task:

- What noises can I make with my lungs, mouth, throat, tongue, lips,...?
- Can I make a combination of noises (a word)?
- Can I repeat particular words on command?
- Can I say words which sound similar to the words I hear adults say?
- What words are associated with particular objects or actions?
- How are words joined together (a sentence)?
- How are words and sentences transformed according to time and context?
- Can I express my thoughts and feelings in accepted sentences?

A possible chain of questions within algebra is:

- Can I find the unknown number, when the list of operations is too long to recall without the help of notation?
- Can I rearrange one equation so that a particular letter/number can end up in a different position relative to the equals sign? How many different positions can it take?
- Can I use my manipulation skills to tackle simultaneous equations?

Successive levels of subordination can help a learner gain fluency with a skill. Once a skill is known, it can also be examined. Vygotsky (1992), in talking about the development of a mental function says that *In order to subject a function to intellectual and volitional control, we must first possess it* (p168). The desire for something to be used and practised before being subjected to conscious attention and examination does not imply that the only method to achieve this is through rote learning without understanding. Sfard and Linchevski (1994) offer a warning about this by discussing a difference between a practising mathematician and a learner of mathematics: *The problem is that unlike the mathematician, the student may easily become addicted to the automatic symbolic manipulations. If not challenged, the pupil may soon reach the point of no return, beyond which what is acceptable only as a temporary way of looking at things will freeze into permanent perspective... It seems very important that we try to motivate our students to active struggle for meaning at every stage of the learning* (p225). Functionalisation offers a way to achieve the practising and using of a new skill through its subordination to a clearly understood task. In this way, a student can work at the task with clear awareness and understanding of what they are doing, it is just that the task involves the frequent use of a new skill in order to achieve that task. This is quite different to the student being told to follow a procedure and repeat it with no clear understanding.

Functionalisation is a powerful notion because it describes so much of the successful learning we have all done. Functionalisation describes the process by which you have been able to develop the skills required to read this paper. Had these skills not been so well subordinated then you would not have had the energy available to engage with the ideas I have attempted to describe within these sentences.

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[1] I am also aware of a similar issues within the USA. For example, there is a Web site in the San Diego region (http://ourworld.compuserve.com:80/homepages/mathman/) where a group are campaigning for traditional teaching methods such as memorisation and drill.

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AN ANALYSIS OF THE DEVELOPMENT OF PUPIL UNDERSTANDING IN GROUP WORK ACTIVITIES USING MULTIMEDIA

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ABSTRACT

This paper offers an analysis of the development of pupil understanding in a group work situation involving classroom activities utilising multimedia. Use was made of the National Curriculum Council sponsored multimedia package "World of Number" (Shell Centre et al, 1993). The study was carried out with a Year 9 (14/15 years) mathematics class working on graphs of relationships between distance, speed and time. The paper is an extension to that presented at PME 19 (Hudson, 1995) in a number of ways. Firstly a fuller analysis of the social interaction within the group is offered. Secondly the interpretation of the development of pupil understanding is informed by insights gained from literature which emphasises the need to move away from a focus on discontinuity, and the notion of "misconception" in particular, towards one which emphasises the importance of continuity in the development from novice to expert knowledge.

INTRODUCTION

The paper begins with an outline of the background to the study and there then follows a full description of the classroom activities and associated resources. My theoretical perspective is outlined in the following section, followed by a description of the methods of data collection and analysis. A framework for the analysis of the resulting social interaction is then considered. There then follow a number of examples of classroom interaction, in the form of video tape transcripts which are analysed in some detail. The development of pupil understanding is considered, with particular regard to one pupil especially. Finally the results of this study are discussed in the concluding section.

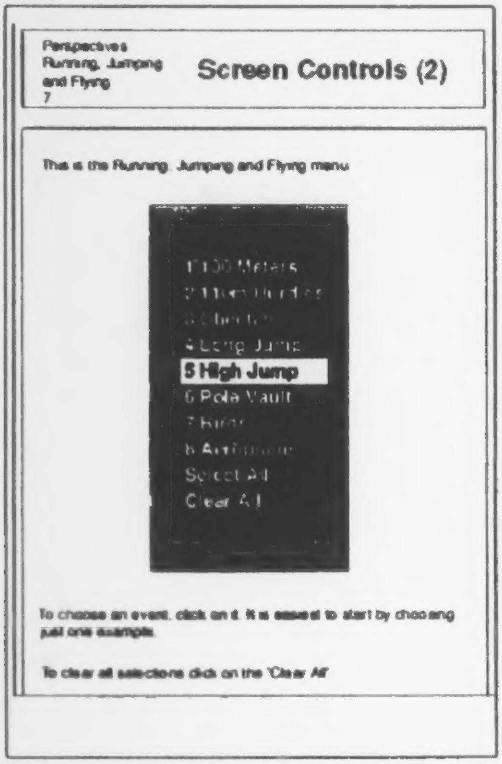
BACKGROUND

The classroom research was conducted as part of a wider project involving the investigation of the potential of group work using multimedia. It was designed to fit in with the planned scheme of work when the group was due to do a two week unit of work on graphical interpretation involving graphs of motion. The topic was introduced as a whole class activity by means of a dice game played in pairs which involved plotting the change of position dependent upon each throw of the dice. The aim of the game was to get to the finish first. Following this activity one of the units from the *World of Number* package was introduced to the whole class with the aim of setting the context and giving the pupils a sense of what to expect in terms of the future activities on the system. The chosen element was the video clip of the women's 100m race in the Seoul Olympics from the unit *Running, Jumping and Flying*. Following the whole class discussion of what the graphs of speed and distance against time might look like some groups began working on the activities at the system

A group size of three had been agreed with the class teacher, with the aim of creating the conditions for effective interaction. Each group was allocated an initial period of thirty minutes for intensive work at the system. The practical limitations were eased considerably by the use of two systems. In addition to the original laser disc package the school also had the use of the CD ROM version. This provision enabled four groups to carry out the multimedia-based activities in a one hour lesson and for each group to have a turn over the period of a single week. The class was timetabled for two lessons of approximately one hour and one of half an hour per week.

MULTIMEDIA-BASED ACTIVITIES

The unit is made up of video clips of various examples of motion, several of which are sporting events from the Seoul Olympics as detailed in Figure 1. Each sequence has two or three graph options associated with it. For example, in the sequence shown in Figure 2, the chosen axes in the bottom left hand window are height and time. Other choices might be distance against time and speed against time. This would give three graphs to choose from in the bottom right hand window. The combined choice is illustrated in the top right hand window.

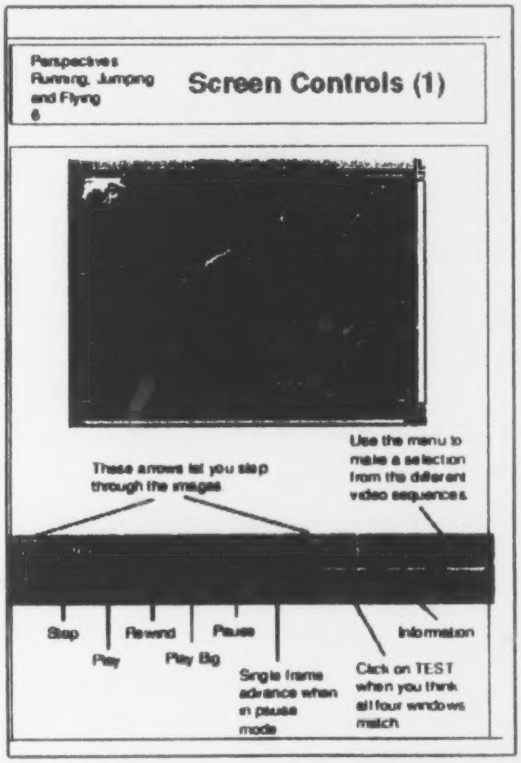

Perspectives
Running, Jumping and Flying
7
Screen Controls (2)
1 100 Meters
2 110m Hurdles
4 Long Jump
5 High Jump
6 Pole Vault
Select All
Clear All

This is the Running, Jumping and Flying menu

To choose an event, click on it. It is easiest to start by choosing just one example.

To clear all selections click on the 'Clear All'

Figure 1


Perspectives
Running, Jumping and Flying
6
Screen Controls (1)
Stop
Play
Rewind
Play Big
Pause
Information

These arrows let you step through the images

Use the menu to make a selection from the different video sequences.

Single frame advance when in pause mode

Click on TEST when you think all four windows match

Figure 2

The main aims of the multimedia-based activity were to promote discussion and provide time for reflection. The activity was structured in such a way as to encourage the following process: select and view a video sequence, think about the distance-time graph, sketch the graph, compare graphs, choose a graph which fits your ideas, explain to each other why a particular graph does or does not

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fit, test out choice on the system and finally repeat the process with a different choice of axes. This can be summarised as a cycle of observation, reflection, recording, discussion and feedback (test), as summarised in Figure 3.

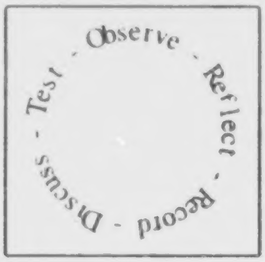


Figure 3

THEORETICAL PERSPECTIVE

The theoretical perspective underpinning this study is based on the work of Vygotsky (1962) and in particular the Vygotskian idea of co-construction as a mechanism for cognitive change. Also of significant influence has been the work of Forman and Cazden (1985) who note that when we try to explore Vygotskian perspectives for education, we immediately confront questions about the role of the student peer group. Forman and Cazden point towards Vygotsky's notion of internalisation, by which the means of social interaction, especially speech, are taken over by the child and internalised and how development proceeds when interpsychological regulation is transformed into intrapsychological regulation. They further highlight the importance of Vygotsky's notion of the *zone of proximal development* and his hypothesis that children would be able to solve problems with assistance from an adult or more capable peer before they could solve them alone.

In Hudson (1995), Vygotsky's notion of the function of egocentric speech is discussed in relation to the development of one pupil's understanding in particular. This analysis is developed further in this paper by drawing upon Confrey's (1995) discussion of the socio-cultural perspective and in particular the dialectic between thought and language, to which she pays particular attention. She outlines Vygotsky's argument that thought and language have different roots. Speech which is the basis for language evolves out of gestures and affective responses whilst thought, and particularly logical thought, evolves from the child's activity and the use of tools.

Vygotsky's notions of spontaneous and scientific concepts are also utilised in this paper. Scientific, or systematic, concepts are seen to originate in schooling whilst spontaneous concepts emerge from the child's own reflections on everyday experience. Spontaneous concepts are seen to work their way upwards towards greater abstractness thus clearing a path for the downward development of scientific concepts towards greater concreteness.

The analysis of the development of pupil understanding is also informed by the work of Smith, DiSessa and Roschelle (1993/94) who suggest that "the fact that students have mathematical and scientific conceptions that are faulty in a variety of contexts can be reframed to highlight their useful and productive nature as well as their limitations". They argue that misconceptions research has focussed on discontinuity although "there is substantial evidence that the form and content of novice and expert knowledge share many features". In support they argue further that expert reasoning involves prior, intuitive knowledge that has been reused or refined and suggest that a fundamental shift is needed in terms of conceptualising knowledge "as a complex system". They agree that the origins of misconceptions lie in prior experience and learning but that conceptions which lead to erroneous conclusions in one context can be quite useful in others. They also note that learning difficult mathematical concepts will never be effortless but also that the support, reuse and refinement of prior knowledge will be an essential prerequisite. They also draw attention to the fact that the notion of replacement of new expert knowledge together with the deletion of faulty misconceptions "oversimplifies the changes involved in learning complex subject matter". Further they draw attention to the fact that misconceptions considered to be extinguished often reappear. They recommend that the goal of teaching should not be to replace misconceptions with expert concepts but rather to "provide the experiential basis for complex and gradual processes of conceptual change".

DATA COLLECTION AND ANALYSIS

The overall approach to the collection and analysis of data was consistent with that outlined by Hamilton and Delamont (1974) who offer an analysis of what they broadly term "anthropological" classroom research. They describe the anthropologist as one who uses an holistic framework, accepts as given the complex scene which is encountered and takes this totality as the data base. There is no attempt to manipulate, control or eliminate variables. At the same time there is no attempt to claim to account for every aspect of this totality in the analysis. A characteristic of the process is that the breadth of the enquiry is gradually reduced to give more attention to the emerging issues. From starting with a wide angle of vision enquiry zooms in and progressively focuses on those classroom features that are considered to be most salient. Thus they argue such an approach clearly dissociates itself from *a priori* reductionism which is characteristic of the more traditional scientific approaches. This approach is also consistent with that of Eisenhart (1988) who considers the relevance of the ethnographic research tradition specifically in relation to mathematics education, and who observes that, central to such an interpretivist approach, is the assumption that all human activity is fundamentally a social and meaning-making activity.

The data was collected by video recording the work of groups working on the multimedia-based activities. The approach to the analysis of the resulting classroom discourse was particularly influenced by the work of Mercer (1991). The focus of the study reported on by Mercer is the content and context of educational discourse in a computer environment from a Vygotskian perspective. The analytic methods adopted are similar to those of ethnography and involved the complete transcription of all the discourse recorded on videotape.

INTERPRETIVE FRAMEWORK

In approaching the analysis of the data arising from the peer interaction, the need for an interpretive framework soon became evident. The approach adopted by Teasley and Roschelle (1993) was found to be particularly resonant and was consequently adapted to form the chosen framework. A framework for the analysis of collaboration is outlined, which the authors argue involves not only a micro-analysis of the content of students' talk, but also how the pragmatic structure of the conversations can result in the construction of shared knowledge. In order to understand how social interaction affects the course of learning, Teasley and Roschelle argue that it requires an understanding of how students use coordinated language and action to establish shared knowledge, to recognise any divergences from shared knowledge as they arise, and to rectify any misunderstandings that impede joint work. The notion of "a shared conception of a problem" is a central one and this is used as the basis of what is described as a *Joint Problem Space*. It is proposed that social interactions in the context of problem solving activity occur in relation to a *Joint Problem Space* (JPS). This is defined as a shared knowledge structure that supports problem solving activity by integrating goals, descriptions of the current problem state, awareness of available problem solving actions and associations that relate goals, features of the current problem state and available actions.

ANALYSIS OF CLASSROOM ACTIVITY

In this episode of classroom interaction Philip, Neil and Jonathan are watching the video sequence of a jumbo jet landing. The axes are initially set on height against time.

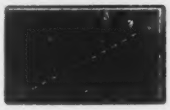
1	P:	It doesn't start off. Watch this. Height against time	Philip runs the video
2	N:	Speed against time that	The axes are set on height against time.
3	P	Yes but no. We've got to choose which height against time is the right one	Trying to clarify the task.
4	J	Let's have a look.	Referring to graph option 1.
5	N	Yah! Oh! How come it does all the wavy lines? It goes straight down It doesn't go up and down does it?	Making a diagonal downward wavy motion.
6	[J	Well change it! Have a look	
7	[P	No but the nose goes up, doesn't it?	Making a diagonal downward smooth motion.
8	[N	No! That's not it!	
9	[J	That's not it!	Referring to graph option 2.
10	N	It's taking off that, isn't it?	

Philip gives a lead at the start of this episode. At line 1, he identifies the problem as being about height against time. However Neil takes his turn by responding to the video with the observation at line 2 that it is "Speed against time that". Philip's response at line 3 appears to be contradictory when he replies "Yes but no". By this he may have been indicating that, "yes", the *graph* showing is the correct choice to fit the speed against time axes but that, "no", it is not addressing the current problem which is "to choose which height against time is the right one." In doing so, Philip seems to be attempting to clarify the task, i.e. to establish the *Joint Problem Space (JPS)*. Jonathan takes his turn to try to move progress with the task itself, when he suggests at line 4 "Let's have a look".

Neil's response to the video sequence at line 5 would seem to be based upon an expectation of a smooth line. However Philip is able to offer an interpretation of the graph, when he observes at line 7 that the nose of the aeroplane "goes up" on landing. The final comment in this section from Neil, at line 10, displays evident confusion between what he interprets from the graph and what he observes by watching the video sequence, which is clearly of the plane landing. The fact that the graph is rising from left to right suggests to Neil that this is the flight path of the aeroplane taking off. This confusion in his thinking was apparent in an earlier episode, when in response to a question about what was happening to the distance covered, Neil's reply was: *"It's going up. Higher"* which was in contrast to Philip who answered: *"It's getting greater"*.

It would seem that Neil's difficulty is related to the fact that he is describing the picture that he sees on the page i.e. "It (*the line*) is going up (*the page*). Higher (*up the page*)". The inability, at this time, to distinguish between the representation of the motion pictorially and the motion itself would explain why Neil interpreted graph 2 as showing the aeroplane taking off.

The next section is later in the same episode when the group is considering distance against time.

11	J:	Do you want to change that one?	Referring to the choice of axes.	
12	P:	Yeh, I've done that. It's distance against time now.		
13	N:	Distance is going down? No! How could it be going down - distance? Oh, it's just landed. But its time's going up!	Referring to graph option 1.	
14	| P:	What?		
15	| J:	The distance? It can't ... can't ...		
16	|	... go down. It just goes up.		
17	N:	I know it can't.		
18	| P:	So, why does it look like that then?	Looking at graph option 2.	

Jonathan's question at line 11 is an attempt to clarify the nature of the task. Philip responds directly by indicating that he has chosen the axes to be considered and elaborates further that "It's distance against time now". Neil's stream of utterances at line 13 seem to form a narration of his current thinking, which once again appears to be very confused. He seeks to interpret the graph in terms of the possible motion of the aeroplane. His first utterance relates to a perception of the distance *going down* rather than decreasing. He seems to dismiss this as a possibility but then refers to the fact that "it" (the plane) has "just landed". He concludes with the utterance "But its time's going up!" without being clear about what "it" refers to.

In response, Philip simply asks "What?", at line 14, and Jonathan attempts to repair the understanding, at line 15, by beginning to suggest that the distance can't decrease. However Neil does not allow him to finish and completes his sentence for him with "... go down. It just goes up." Although this completion is distributed over a single sentence, there is evident conflict within the group in terms of their shared understanding. Philip intervenes at line 17 and asserts that "I know it can't (go down)" which elicits the question from Neil "So, why does it look like that then?".

DISCUSSION

In this episode Philip gives a lead on a number of occasions which take the form of clarifying the problem or establishing the *JPS*. He does this at the start of the episode (lines 1 and 3) and also later at line 12 when the axes had been changed to distance against time. In doing so, he is assisted by Jonathan who, for example, at line 11 asks *"Do you want to change that one?"*. Jonathan is also influential in moving the group forward when he suggests *"Let's have a look"* and also *"Well change it! Have a look ..."* at lines 4 and 6. By contrast, Neil's utterances are based on his reactions towards what he sees on the screen. He is also much less clear about what it is that he is describing. For example at line 5, he seems to use *"it"* to refer to two things without being clear about the distinction between them i.e. the aeroplane and the representation of its path in the form of the graph. He is again unclear at line 13 in terms of what *"it"* refers to. In fact Philip responds directly to Neil's question at line 18 by stating that *"it starts from the bottom and goes up"*. In doing so, Philip is clearly referring to the graph although this is not explicitly stated. He subsequently asserts that *"It's got to be that"* and appears to be quite certain. This graph option is the only one which *"starts"* at the origin or in Philip's words *"starts from the bottom and goes up"*. Neil is also convinced by the time the group comes to test out their choice on the system and exclaims that *"It is right. Because the distance goes up and so does the time, at the same time!"*.

In Hudson (1995) Neil's confusion was described in terms of a misconception. However as Smith, DiSessa and Roschelle (1993/94) observe, in emphasising the discontinuity between novice and expert knowledge the potentially "useful and productive nature" of such can be lost sight of. Initially Neil does not display confusion of a significant nature but merely asks why *"it does all those wavy lines"*. On first seeing graph option 2, he is quite sure that it is not the correct graph and asserts at line 8 *"No! That's not it!"*. Subsequently he reacts to the graph by seeking to interpret it in terms of the potential motion of the plane and it is at this stage that the idea of the plane taking off is introduced. He again responds to the graph showing on screen at line 13, with a stream of utterances which display considerable confusion on his part.

As discussed in Hudson (1995), Neil's use of language is resonant with Vygotsky's notion of *egocentric* speech. In highlighting the dialectic between thought and language Confrey (1995) draws particular attention to Vygotsky's argument that these have different roots and hence that there are two distinct lines of development which eventually lead to a synthesis. Vygotsky proposed that speech can be considered to have two particular forms which he describes as egocentric and communicative respectively. The function of communicative speech is for the purpose of communication with others whilst the function of egocentric speech is as an instrument of thought itself. Vygotsky develops this view of the function of egocentric speech, by arguing that all silent thinking is "nothing but egocentric speech". Vygotsky also notes that children resort to egocentric speech when faced with difficult situations. He argues further that egocentric speech is the genetic link in the transition between vocal and inner speech.

Many of Neil's utterances are consistent with egocentric speech, in contrast to both Philip and Jonathan whose utterances seem to follow from their own reflections on the situation. Neil's level of achievement was in fact one of the lowest in the class and his performance on the delayed post-test was slightly worse than on his pre-test. These results suggest that Neil's confusion deepened

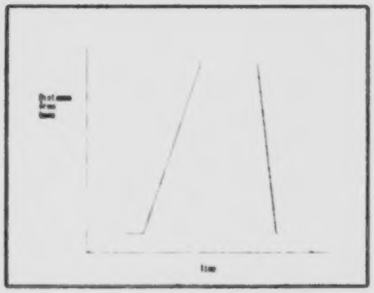


Figure 4

over the course of time and this is illustrated by his response to being asked to describe a story to fit the graph in Figure 4 (the axes are *distance from home* against *time)*. Neil's response was:

> *"An aeroplane coming out of a hangar and getting on to the runway it pauses for a little while and hovers forward into the air but stops for a while then it comes back down again."*

This interpretation completely ignores the fact that the graph is of distance against time and not of height against time. However for a graph of height against time it would be one possible interpretation. Further it suggests that Neil's spontaneous conception is of a graph of height against time. This is a situation with which he is comfortable in which he is utilising prior, intuitive knowledge - although not necessarily answering the question put. The real difficulty for Neil seems to arise when the transfer from the motion to its graphical representation is not in the corresponding plane, which appears to be the case with height but not with speed and distance. This example is an illustration of how misconceptions which are considered to have been extinguished often reappear and it also highlights the need to provide pupils such as Neil with further experiences as the basis for "complex and gradual processes of conceptual change" (Smith, DiSessa and Roschelle, 1993/94).

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SOME ISSUES IN ASSESSING PROCEPTUAL UNDERSTANDING

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This paper reports on one strand of ongoing research on 13-15 year old students' understanding of algebraic expressions, that of students' proceptual understanding. In preparation for a larger study tools for assessing students' proceptual understanding of algebraic expressions through test items were developed. Students who had shown evidence of proceptual understanding were subsequently interviewed to evaluate these written tools. A number of issues surrounding the assessment of proceptual understanding are discussed including the method of assessment, the context of the question and students' technical skills. The extent to which conceptions can be ascertained from test items alone is discussed.

Introduction

Research is currently being conducted to determine the effects of a cognitive conflict, intensive discussion approach to teaching and learning algebra (see Bell, 1986). This research requires an understanding and categorisation of students' concepts surrounding the use of letters and expressions (see Perso, 1991). There are two main areas of investigation, the concept the student has of letters within expressions and the strategies and errors involved when working with expressions. However, there is a third, subsidiary, but interesting area of understanding – that of the expression as a whole. This brings in the ideas of understanding the expression as representing a procedure to be carried out, or as a structural object (concept) that can be manipulated. PME has been an important forum for developing these ideas which Sfard (1989) calls 'structural and operational' and Gray & Tall (1991) call 'proceptual'. We use Gray & Tall's terminology here.

> We shall refer to the combination of process and concept represented by the same symbol by the portmanteau name 'procept'. (*ibid*, p.73)

Why have we chosen to focus this research on students' understanding of letters in expressions rather than equations? The issues surrounding students' understanding of letters are, on the whole, similar in both equations and expressions. However, the study of equations tends to bring in several features that add to students' difficulties without necessarily shedding much light on their understanding of letters. For example, when students have to solve equations, there are a number of procedural as well as conceptual difficulties that need to be overcome. With a study focusing on expressions we hope it will be easier to isolate the conceptual difficulties from the procedural ones.

As part of the research methodology, groups of students were given a diagnostic test. Test items were designed to elicit students' conceptual understanding of letters within expressions and their strategies for working with letters in expressions. Other test items were designed to provide information on students' proceptual understanding

This paper discusses some of the issues surrounding the notion of procepts and the categorisation of students as having a proceptual perspective. The research reported here is in its pilot phase and is working towards a classification scheme that allows students' responses to be categorised as procedural, conceptual or proceptual. Various 'working criteria' couched in terms such as 'two or more different but correct answers', 'reasonable justification other than *the same but written differently*' and 'ignore purely spatial rearrangements of terms' were considered.

Methodological issues

There are a variety of methods that can be used in order to glean some evidence about the way in which students think. However, as Sfard and Linchevski (1994) point out,

> "a painstakingly detailed scrutiny of student's behaviours and utterances ... is necessary to have some insight into his or her thinking". (pp.192-193)

Our research involved giving 80 students a diagnostic test aimed at obtaining information on their understanding of letters and the strategies and errors observed. Within the context of the wider research a detailed scrutiny of proceptual understanding was impractical. So what value can we place on responses to test items designed to elicit students' proceptual understanding?

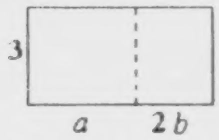
A test will only give the conclusions of students' thought process and does not give much, if any, evidence about the way they arrived at their written response. To provide further evidence to support the written outcome, a small sample of the students that took the test were interviewed about responses that appeared to show a proceptual understanding. Students were asked to justify their previous answers and answer some related questions during the interview. One of the issues that was explored in the interviews was the part placed by the context of the question. Some questions used diagrams to act as a visual representation of the algebraic expression – did students reach their answer through the medium of the diagram rather than from consideration of the algebraic expression alone?

Test items

Only those items relating to proceptual issues are presented here. These items were used as a basis for the subsequent interviews

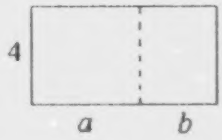
Question 1

You can write down the area of this rectangle as $3a + 6b$. Write down as many other expressions as you can for the area of this rectangle.



Question 2

You can write down the area of this rectangle as $4(a + b)$. Write down as many other expressions as you can for the area of this rectangle.



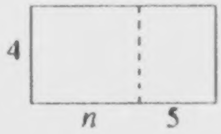
Question 3

a) What does $x + 2 + x + 2$ give when $x = 6$? b) What does $2x + 4$ give when $x = 6$?
c) What does $4 + 2x$ give when $x = 6$? d) What does $2(x + 2)$ give when $x = 6$

Explain how the answers are linked. Explain how the expressions are linked.

Question 4

You can write down the area of this rectangle as $n + 5$ multiplied by 4. Write down as many other expressions as you can for the area of this rectangle.



Students' responses to the test items

The responses given below are from eight 14-15 year old students who were interviewed after having their test responses were analysed. These students were selected as a sample from those students who showed some evidence of a proceptual perspective on one or more of the four questions above. The test and protocol data described below is representative of the different types of responses that the students gave during the test and the interview.

The difficulty of defining what a 'concept' is leads to difficulties in defining a procept. This, however, is compounded when assessing proceptual understanding through test items. As students with proceptual understanding will not only be flexible about which perspective they are using, but also ambiguous. The key aspect of students' work that needs to be seen is that they can use an expression conceptually.

Question 1 asked students to give expressions that are equivalent to $3a + 6b$. Student S gave four expressions on the test

$3(a + 2b)$ $\quad a + a + a + 6b$

$3a + 2b + 2b + 2b$ $\quad a + a + a + 2b + 2b + 2b$

This set of responses shows that student S can represent the same expression in a number of different forms. Student S was then interviewed and explained her responses using the rectangle, but perhaps only as a 'prop', as above. S was then presented with a further rectangle and gave the answers $6xa + 2b$, $6(a + 2b)$, $8(b + a)$. The last answer here was derived from adding all the numbers round the rectangle (3, 3, and 2). This latter response shows the difficulty of any form of assessment, that a student might respond correctly on one occasion and incorrectly on another.

Question 2 requires the students to give as many algebraically equivalent answers as possible to $4(a + b)$. Student S gave the following answers on the test:

$a + b4$ $\qquad$ $a + b + a + b + a + b + a + b$ $\qquad$ $a + a + a + a + b + b + b + b$.

The first answer shows a misunderstanding of the use of brackets, but do the next two answers show a proceptual understanding? When interviewed about the question, S explained that $a + b4$ meant 4 times a and b. This seems to show a flexibility of understanding, but not one that is generally accepted! The issue here is whether technical errors alter students' proceptual perspective.

Student J seems to show a clear conceptual understanding of $4(a + b)$ on the test, as the following answers were given as equivalent

$4a + 4b$ $\qquad$ $a + a + a + a + b + b + b + b$

$4a + b + b + b + b$ $\qquad$ $4b + a + a + a + a$.

However, when interviewed, J said "they all equal the square", indicating that the understanding was perhaps more due to the diagram used to give the question greater meaning. J also went on to say that the expressions were "the same answer but different ways of writing it". Is this simply the use of 'answer' to mean 'expression' or has the question been understood as implying a numerical result? During the interview, a follow-up question was asked that did not use the rectangle as a context or representation for the algebraic expression. In this case, J wrote

$3x + 6y$ $\qquad$ $x + x + x + y + y + y + y + y + y$

$x + x + x + 6y$ $\qquad$ $3x + y + y + y + y + y + y$

as equivalent to $3(x + 2y)$. This shows that, although the explanation during the interview implied that the diagram had been used, it seems that this may have been more an aid to explanation rather than an aid to understanding.

This use of the diagram was even clearer with student M, who explained in the interview that "I multiplied the first box, well I found the area of the first box by timesing that length by 4 and I did the second one and I added them together...". However, when asked a similar question without the diagram, the student could still explain that $3(x + y)$ was equivalent to $3x + 3y$ and $x + x + x + y + y + y$.

Question 3 was a different style of question and first asked for an explanation of why the answers to each part of the question were (or should have been!) the same. The second part asked why the expressions were the same, with the first part of the question being used to try to avoid students simply saying that the answers were the same.

Student T gave the answers to the first part of the test item as a) 16, b) 30, c) 30, d) 16 (the answers of 30 were arrived at by considering $2x$ to be 26 when $x = 6$, a 'category one' type error according to Perso (1991, p.11)). The student then wrote down that 'The expressions for the answer 30 are the same but in a different way and the same for the answer 16'. Whilst $2x + 4$ and $4 + 2x$ can be seen to be 'the same but in a different way', was this how the expressions $x + 2 + x + 2$ and $2(x + 2)$ were seen, or is the last part of the students explanation merely an afterthought following from the fact that the answers were equal?

This was followed up in the interview ('I' being the interviewer).

T For the 16 that just says x add 2 add x add 2 it is just saying that 2 times x add 2 is the same answer.

I So the answers are the same, what about this algebra… Can you tell me they were the same from there?

T Yes, if you add brackets in, that will be … 2 times x plus 2.

From a not very promising start, which seemed to show that the student had only compared the answers, T then gave a reasonable algebraic justification why the two expressions were equivalent.

After correctly evaluating each of the expressions on the test as being 16, student L explained that the answers are linked "because it's the same question only put out different", and explained that the expressions are linked "because it means the same thing". When interviewed, the student gave the following responses.

L: They all add up to the same thing.

I Can you look at the letters and numbers and tell me, can you see a connection between the letters and numbers as well?

L: No reply.

This student seems to have viewed the expressions in a purely procedural way.

Question 4 again asked students to give alternative expressions for $n + 5$ multiplied by 4 with a split rectangle displayed. Student M gave the following answers

$(5 + n)4$ $5 + 5 + 5 + 5 + n + n + n + n$.

These were then discussed during the interview.

M: Well I added those two lengths together first so it is 5 plus n then I timesed them by 4 and then I wrote down 5 plus 5 four times and I added n and I wrote that down 4 times.

I Can you write an expression that means the same as x plus 4 multiplied by 3?

M: $3(x + 4)$ $3x4 + 3x$

From the interview, the student seems to be making use of the diagram, although the second expression is not related to the diagram at all and the student is capable of working without the diagram at all.

Discussion

One problem we anticipated was the difficulty of assigning meaning to an expression such as $3(a+2b)$ when no numeric substitution was required. The expression may appear as an abstraction with no immediate reference (no 'sense' and no 'reference' in Fregian terms). This is why we initially inserted the split rectangle in questions 1, 2 and 4 – intuitively, at the time, to make it accessible to the students. In retrospect we believe our motive was to give the expression 'meaning' by providing a reference. But in doing this we have made our proceptual analysis of students' responses more difficult because the figure may convey both process and product – the figure may present both one rectangle and two rectangles forming a sum rectangle. For many learners it will, subconsciously, do both things at once, i.e. it will present a flexible and ambiguous procept whereas a similar question without a figure appears less likely to prompt both process and product. Both forms present representations of an algebraic procept. But

> "A representation does not represent by itself - it needs interpreting and, to be interpreted, it needs an interpreter." (von Glaserfeld, 1987).

But how do we interpret the interpretation of the interpreter? Further there are many forms of representation (*ibid*). The figure appears as an (indirect) *iconic* representation. Piaget's distinction between *figurative* (relating to the observable) and *operative* (relating to inference) knowledge has relevance here (see Furth, 1977). But, as Furth makes clear, the distinction is problematic for what is observed is a function of what the learner already knows. This problem is evident when we attempt to analyse the protocols. Student S in question 1 explained her responses in terms of the rectangle but perhaps only as a 'prop'. Student J, we felt, may have been using the rectangle as a vehicle to assist the explanation to us. Student M focused on the rectangle initially but later 'appeared' to function without reference to it.

A problem that we have with procepts is with concepts. Skemp (1971, p.27) states that "concept itself cannot be defined" but notes that we may "describe some of the characteristics of concepts". This is not an argument in itself against the term *procept* but should cause us to take care with the term. It may also partially explain why some people find they 'cannot get a handle' on procepts. Dubinsky (1991) and Sfard (1991) by-pass some of these problems by speaking of process-object rather than process-concept. Reflecting again on our intuitive intentions in the wider research related to categories and hierarchies of students' conceptions of expressions, we happily speak to each other of students' concepts and their conceptual understanding but experience difficulty when called upon to explain this. Further, as the two interview extracts from student T suggest, some students appear to have a conceptual understanding that is grounded in handling expressions in a procedural manner.

Let us take this theme a little further for it relates to difficulties in developing criteria for categorising students' responses as showing proceptual understanding or not. Following Fischbein *et al.* (1979) we note the labile nature of some students' conceptions. An example of this is the response of student S to question 1. Here the student seems quite capable of a conceptual response on the test. However, when interviewed and given a very similar question, also with a diagram, the student responded in a way that was not obviously proceptual. In this example, the student's understanding seems to have changed between the test and the interview. The reasons for this may be manifold, but it seems that some important variables include the method used to assess the student's understanding, the context of the question and the student's grasp of the concept itself. The student may have been more able to reflect on the question in a test, but felt under pressure to put down a rapid answer in an interview. The diagram was slightly more complex in the interview and this may have meant that, whilst the student had the understanding necessary to cope in a certain set of situations, this understanding was no longer sufficient for the new situation. It is also important to consider that when a student has partially attained an understanding of a concept this may be exhibited by a conceptually correct response on one occasion and an incorrect response on another, even on identical questions.

A related difficulty in assessing students' understanding from a test is the part played by the student's technical ability. This is shown by student S on question 2 where the first answer is given as $a + b4$. This is algebraically incorrect, although the student explained during the interview that this expression meant 4 times a and b. The student's meaning on the test item is hidden by a technical error, but from the interview is seems that the student does have the conceptual understanding. In a similar way, student T's response to question 3 contains a place value error when substituting $x = 6$ into two of the expressions. This student then went on to give an almost 'textbook' explanation of why two expressions are equivalent, but a similar technical error *could* have resulted in the equivalence of the expressions being overlooked by the student.

Conclusion

As in many situations, a single administration of an assessment item reveals very little of a student's underlying understanding. The responses are affected by many external (to the student) factors such as the context of the question and the method used to assess their understanding. There are also barriers between what the student understands and what the student writes down or says. In the case of a test, technical errors in what is written down may mask a conceptual understanding either to the student or to the assessor or both. Students who are just developing a concept that is being assessed may sometimes exhibit the concept and sometimes not.

The issue of technical errors masking the assessment of conceptual understanding is an important one. Students may still understand the concept even if there are errors in their technique, for example with the consistent use of brackets. However, students' ability to express their thoughts to others are impaired, and our ability to interpret them is much impeded, without extensive follow up work.

Procepts are a very important construct for work in mathematics education but little in the existing literature prepared us for the problems we experienced classifying students' understanding in terms of procedural, conceptual and proceptual understanding. Perhaps too much emphasis by writers is put on theoretical issues and not enough on analysing real students responses? The importance of the notion may also lead us to ascribe too much to proceptual analysis. Our discussion of forms of representation lead us to believe that procepts are only part of the picture.

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THE DEVELOPMENT OF LANGUAGE ABOUT FUNCTION: AN APPLICATION OF VAN HIELE'S LEVELS

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ABSTRACT

This paper proposes a model of the development of language about function. This model was developed by comparing Japanese teaching practices and national curriculum with generalized forms of van Hiele's Levels. This paper points out features of van Hiele's Levels and shows that they are also characteristics of the proposed levels of language about functions. These features include: language hierarchy, the existence of un-translatable concepts, a duality of object and method, and mathematical language and student thinking in context. The levels indicate that students' development resembles an expanding equilibration, rather than a monotonous increase of knowledge.

Introduction

In the past ten years, multi-representational tools for exploring functions have been changing the contexts and learning sequence of arithmetic, pre-algebra, algebra, pre-calculus and calculus. In discussing these reforms, it is important to consider students' development not only in terms of conceptual functional thinking but also considering students' language concerning functions. Several models of the development of functional thinking have been proposed (E. Dubinsky, 1992; A. Sfard, 1991; A. Sierpinska, 1992; S. Vinner, 1991). These models' views imply that the development of students' knowledge and thinking about function is like an expanding equilibration rather than a monotonous increase (cf. J. Confrey 1994; E.V. Glasersfeld 1995). This paper will show another model of development which provides the same view but focuses on the students' development concerning the representations of function as mathematical language. One characteristic of this model is its background. This model was set by comparing the Japanese national curriculum and teaching practices with the generalized forms of van Hiele's Levels (A. Hoffer 1983; M. Isoda 1984). The Japanese curriculum may be the only national curriculum which has specified areas of function/functional thinking from elementary school.[1] This paper discusses the features of this model from the view point of expanding equilibration and the features of van Hiele's Levels.

The Features of van Hiele Levels

A. Language Hierarchy. Each level has its own language and the levels are hierarchical (van Hiele, 1959).

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[1] In the national curriculum, the 4 areas of elementary school and 5 areas of junior high school mathematics have been formally in place since 1958. These areas include functional thinking, figures/geometry, and arithmetic/algebra. Each area is connected and integrated with each other. This tradition has its roots in the movement of Perry, Kline & Moore.

B. Existence of Un-translatable Concepts. The corresponding contents of different levels sometimes conflict (van Hiele, 1986).

C. Duality of Object and Method. The thinking of each level has its own inquiring object (subject matter) and inquiring method (the way of learning). The method of each level is verbalized and becomes the object, subject matter, of the next level's inquiring. This is the duality between object and method (van Hiele 1958; H. Freudenthal 1973; I. Hirabayashi 1978).

D. Mathematical Language and Student Thinking in Context. While the levels are distinguished as sets of mathematical language, the actual thinking of each student varies depending on the teaching and learning context (van Hiele, 1958; M. Isoda, 1988; D.Clements, 1992; cf. M. Battista, 1994).

The last feature claims that we should make a distinction between the levels of mathematical language and the levels of students thinking itself. Although several research studies have attempted to evaluate individual student's levels of geometric thinking, they point out the difficulty in doing so (cf. J. Mayberry, 1983; A. Gutiérrez, 1991; D.Clements, 1992). If these four features can be pointed out in another area of mathematics, for example the area termed 'functional relation' in Japan, we could conclude that it is an application of van Hiele's Levels. This paper first discusses the levels of function from the viewpoint of language and then discusses the development of students' skills.

The Levels of Language about Functions

Through investigations[2] of the development of students' language for describing functions and the history of the description of motion, the following levels of function have been discussed[3,4] (M. Isoda, 1987, 1988, 1990). Historical examples are written in footnote five through eleven.

Level 1. Level of Everyday Language.

Students describe relations in phenomena using everyday language obscurely. They can discuss changes in numbers using calculations, but usually their descriptions are done with or focused on one physically evident variable, the *dependent variable*.[5] Even if they are aware of covariation, it is difficult for them to explain it appropriately using two variables because their descriptions of relations are done obscurely[6] using everyday language . So it is difficult for them to compare different phenomena at once, appropriately.

Level 2. Level of Arithmetic

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[2] Investigations included tests, interviews and teaching practice/classroom observations.

[3] In van Hiele theory, levels are described with like these generalized students' activities. But these generalized description are already mentioned the level of language rather than each student's thinking itself. Because depending on the context/educational situation, students could do more higher level activity and students' activity usually included lower level activity and change depending on context.

[4] Because curriculum and students' development are mutually related, students' development reflects the curriculum and investigations of development cannot prove its hierarchy. Phylogenetic examples are a good ground for ontogenesis.

[5] Zeno, Eleatic school, argued that Achilles could not catch a tortoise.

[6] Aristotle wrote that something falls faster if it is heavier

Students describe the rules of relations using *tables*. They make and explore tables with arithmetic. Their descriptions of relations in phenomena are more precise with tables than with the only everyday language of Level 1.[7] Students have general concepts about some rules of relations[8], for instance, *proportion*. Students can compare different phenomena using such rules. They describe rules of relations as covariation and when reading tables, their interpretation of the covariation of variables is at least as strong as their interpretation of correspondence. Students can use formulas and graphs to represent rules and relations, too, but it is not easy for them to translate between notations.

Level 3. Level of Algebra and Geometry

Students describe functions using *equations* and *graphs*. To explore function, they translate among the notations of tables, equations and graphs and use algebra and geometry.[9] At this level, their notion of function, which they already understand well, involves the representation of different notations already integrated as the mental image. For example, they can easily find the equation emerging from the graph, and the graph from the equation.

Level 4. Level of Calculus

Students describe function using calculus. In calculus, functions are described in terms of *derived* or *primitive functions*.[10] For example, to describe the features of a function we use its derived function which is already learned. The theory of calculus is a generalized theory of this type of description.

Level 5. Level of Analysis

An example of language for description is functional analysis which is a metatheory of calculus. This level's justification is based on historical development[11] and not yet investigated.

Table 1 shows the duality between object and method in van Hiele's Levels (the Levels of Geometry) and in the Levels of Function. Examples of untranslatable concepts are offered between each level. Furthermore, the existence of duality and untranslatable concepts suggests a hierarchical relationship between the levels.[12] Thus, these constitute three of the four features of van Hiele's Levels listed earlier in the paper. These, as well as the fourth feature, will be further discussed later from the viewpoint of the development of student thinking.

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[7] Ptolemy made the chord (trigonometry) table to describe the motion of planets.

[8] Galileo found the ratio of differences in the distance fallen of falling bodies to be the sequence of odd numbers.

[9] Galileo found the parabola, which Apollonius had described as being cut from a conic, from the odd number ratio of a falling body.

[10] Newton descried motion using fluxion.

[11] J. Bernoulli posed the problems of branchistochrone and geodesic line. These variational problems were origin of functional analysis and differential geometry.

[12] In the case of quadratic function, we can make the following distinctions.

Level 1. Students do not easily compare the situations. They can not appropriately distinguish quadratic from other situations if we use only daily language. See footnote No. 6.

Level 2. Quadratic functions and contextual situations can be described using a table where second differences are constant.

Level 3; Quadratic functions are described algebraically by $y=ax^2+bx+c$, and geometrically by parabolas. Tangent lines are discussed using b^2-4ac

Level 4; Quadratic function is described with the derived function of cubic function and primitive function of linear function. Tangent lines are discussed using derivative

Table 1. Duality and Un-translatable Concepts

	The Levels of Geometry	The Levels of Function
Level 1	Students explore matter (object) using *figures* (*method*).	Students explore phenomena (object) using obscure *relations* or *variation* (*method*).
Example of conflicts between levels	Because it has rounded corners, the road sign 'YIELD' is not a triangle according to the meanings of Level 2, but we call it a triangle in daily language.	In Japanese, we use "2 BAI, 3 BAI" to mean "two times, three times" on level 2. But in everyday Japanese (Level 1), we can use "BAI" to mean either "double" or "plus". A child on level 1 says "BAI,BAI" ("plus plus") to mean three times the original amount. But "BAI,BAI" ("double double") usually means four times. On level 2, students use "2 BAI, 3 BAI" to explain proportion as a covariance and they say three times as "3 BAI" and do not say it "BAI,BAI".
Level 2	Students explore the figures using the *property*; The object on level 2 was the method on level 1.	Students explore the relations using *rules*; The object on level 2 was the *method* on level 1.
Example of conflicts	A square is rectangular on Level 3, but not on Level 2.	The constant function is a function on Level 3 but 'constant' is not the relation which was discussed as covariation on level 2.
Level 3	Students explore the properties of figures using *implication*.	Students explore the rules using *notations of functions*.
Example of conflicts	The isosceles triangle has congruent angles. On Level 3, it is induced already and we do not have to explain more. On Level 4, we prove it.	On Level 3, a tangent line of quadrilateral function deduced using the property of only one common point / a multiple root. On the Level 4, the tangent line does not always have this property.
Level 4	Students explore the proposition, which is formed by implication, using *proof*.	Students explore functions using *derived or primitive function*.

The Development of Students' Thinking

Students' development from a lower level to a higher level resembles an expanding equilibration rather than a monotonous increase. Below, two examples are offered which were selected from investigations of the development of functional language from level 2 to level 3. The features of van Hiele's Levels help explain the students' growth of knowledge. First, I describe the Japanese curriculum for moving from level 2 to level 3.

In the national curriculum in Japan, an informal notion of proportion is taught in grade 4 and more formal concepts of whole number proportion including $y=ax$ are taught via real situations in grade 6. The curriculums of grades 4 through 6 are regarded as level 2 or as a transition to level 2. In grade 7 (junior high school grade 1 in Japan), students learn how to solve equations with one variable, the definition of function using the idea of correspondence, and the function $y=ax$. In grade 8, the linear function $y=ax+b$ is taught. In grade 9, the quadratic function $y=ax^2$ is taught and function is redefined using the idea of set and correspondence[13]. The curriculums of grades 7 through 9 are regarded as level 3 or as a transition to level 3. The investigation found that many students in grade 6 thought on level 2, and many in grade 10 thought on level 3.

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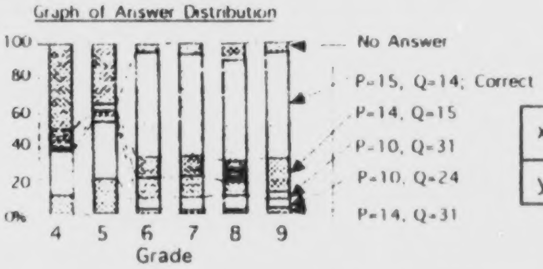
[13] In the current curriculum, this redefinition of function is taught in Grade 10. Examples were collected in the former curriculum.

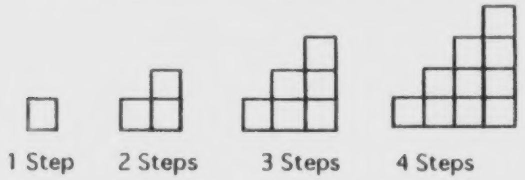
Example 1. Students lose their connection to lower levels of thinking in the process of moving to a higher level. The results[14] of problem 1[15], below, show that students' proportional reasoning looks the same[16] after they learned the formal concept of proportion via situations in grade 6 and after they re-learned the concept as the function y=ax in grade 7. But the results of problem 2 show the change in their reasoning from grade 6 to grade 7. Q3 in problem 2, (see Graph 4) shows that grade 6 students' proportion of correct answers was higher[17] than in grade 7, but is the same as grade 9. Graph 5 indicates that, to get a correct answer, grade 6 students' solving methods of problem 1 and of Q3 were more different than grade 7 students'. Q2 in problem 2, (see Graph 3) shows that many grade 7 students still recognized this situation as dealing with proportions. Graph 6 show that half of them could not write a correct answer to Q3. The difference between problem 1 and problem 2 is that problem 2 was posed via a real situation. This result suggests that many grade 7 students, in the process of reconstructing the concept of proportion as a function, become lost when applying the concept of proportion to the real world. Indeed, Graphs 1 and 2 for Q1 show that after learning proportion, grade 6 students could describe and analyze the situation itself exactly, while grade 7 students, having re-learned proportion as a function, could not.

Problem 1 In the right table, if y is in proportion to x, then select the pair which is appropriate for P and Q in the table.



x	3	6	P
y	7	Q	35

Problem 2. Let's make stairs using squares with sides 1 cm as follows.



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[14] This data was collected in a down town area of big cities and each grade's population was larger than 150 people. They had already learned each grade content of function or functional thinking area in the national curriculum.

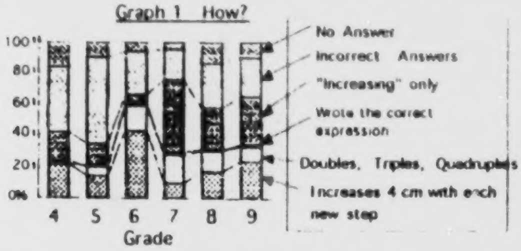
[15] This problem is the same as a problem in the Second International Mathematics Study.

[16] The probability of no difference is 0.6. There is no significant difference.

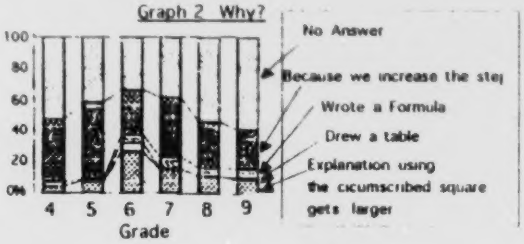
[17] The probability of no difference is 0.00015. There is a significant difference.

Q1. How does the perimeter change as the number of steps increases? Why do you think so?

Graph 1 How?

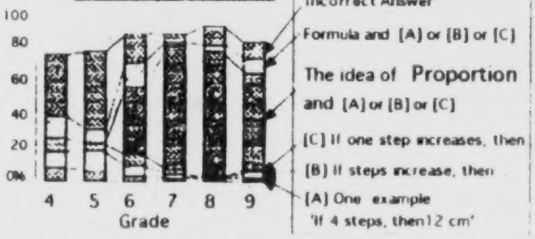


Graph 2 Why?

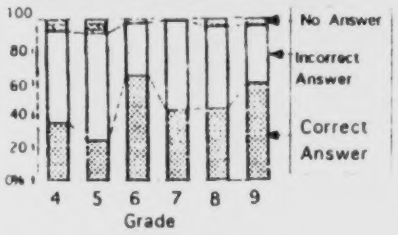


Q2. How can we relate the number of steps and the perimeter?
Q3. What is the perimeter if there are ten steps?

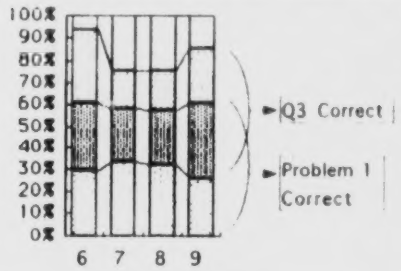
Graph 3. (Q2) Relation



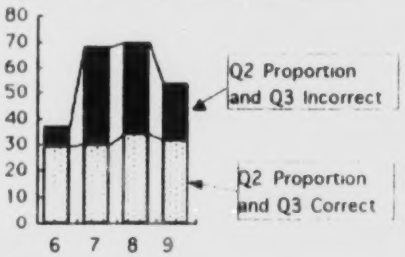
Graph 4. (Q3) Application of Relation



Graph 5. Cross-Analysis of Problem 1 and Q3



Graph 6. Cross-Analysis of Q2 and Q3



One interpretation of these results is that many students who already know the concepts of proportion and have experience dealing with y=ax in the context of real situations, can not assimilate the function y=ax in the context of algebraic discussions. But Q3 of problem 2 shows that, in grade 9, many students are again able to find the answer. Thus, it can be interpreted that students in grade 9 had accommodated their knowledge.

Example 2. Students' thinking is still viable until they meet a non-viable situation; We read tables as representing covariation and correspondence. In the Japanese curriculum, functions are taught using correspondence in grade 7 to assist students to level 3. Teachers begin to call a table of function a 'Correspondence Table' when teaching correspondence. But the results of problem 3 show that students do not change their thinking until grade 9 during which they learn the function $y=ax^2$, which is not easy to read covariationally. Indeed, in spite of students having been taught the y=ax table as correspondence from grade 7, many students continued to read the table covariationally until they were taught $y=ax^2$.

Problem 3. Write what you can find from the following tables.

(1)

x	1	2	3	4
y	4	8	12	16

(2)

x	1	2	3	4
y	2	8	18	32

Result of (1)

Grade	6	7	8	9
Covariation	42%	48%	49%	35%
Correspondence	24%	14%	16%	35%
Both	8%	10%	11%	11%

Result of (2)

Grade	6	7	8	9
Covariation	27%	15%	11%	10%
Correspondence	37%	21%	20%	50%
Both	0	0	2%	3%

Discussion

Examples 1 and 2 show that teaching supports students' transitions to level 3. It would be better to interpret the development of students' thinking from a lower level to a higher level as resembling an expanding equilibration rather than a monotonous increase. Furthermore, the above examples reflect the features of van Heile's Levels. Indeed, based on these features, we can more critically interpret these examples.

Critical Explanation of Example 1. In order to explain example 1, hierarchy and duality of the levels of function in the context of Japanese curriculum must be discussed. To move students to level 2, teachers teach rules, for example proportion, using arithmetic on tables via real situations which were represented on level 1 using everyday language. To move students to level 3, teachers teach functions using algebra and geometry via rules which were represented on level 2 using arithmetic language with tables. Arithmetic language claims to move students to level 3, but in the case of everyday language, although students use it, they do not need to use everyday language in order to learn about functions algebraically and geometrically.

The notions of hierarchy and duality support a clearer explanation of example 1. Indeed, in grade 6, to move to level 2, teachers teach the concept of proportion using tables via real situations on level 1. And in grade 7, to move to level 3, teachers teach functions of the form y=ax using equations and graphs via the concept of proportion which was represented in arithmetic tables. Therefore, in problem 1, which was only represented with a table, there is no difference between the results in grade 6, 7, 8 and 9. But problem 2 was represented with a situation. Because grade 7 students had not learned the *function* y=ax with situations using everyday language, they overlooked/lost proportional reasoning in the situation.

Critical Explanation of Example 2. If we suppose that student thinking can be changed depending on the context of the teaching situation, example 2 can be more fully explained. Despite the fact that teachers explain correspondence using tables of y=ax, the students did not change their reasoning. But when teachers explained correspondence on tables of $y=ax^2$, the students did change their reasoning. Indeed, in the case of the table for y=ax, students learned covariance in grade 6 (level 2), and as we already saw in example 1, it was not changed in grade 7. If they know covariance, e.g. the that first difference of y=ax and y=ax+b is constant, then they can make a table. So, this knowledge is still viable in grades 7 and 8, during which they move to level 3. But in the case of $y=ax^2$ taught in grade 9, the first difference is not constant. In order to make a table, since students could not use the first difference they had to use correspondence. Thus, the quadratic function $y=ax^2$ provided a context that helped students understand the notion of correspondence.

Table 1, Examples 1 and 2 indicate that the levels of function include all four features of van Heile's Levels. Furthermore, it has been implyed that students' thinking is better characterized as an expanding equilibration

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HITOMI'S MEANING CONSTRUCTION OF TABLE AND ALGEBRAIC EXPRESSION OF PROPORTION DURING INSTRUCTION: A CASE STUDY

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This article documented one sixth grade student's interactions with notations of tables and $y=mx$ during the class work in proportion, which was deeply interwoven with her proportional reasoning. Naturalistic method was used in order to get a thick description. The student was observed to have developed different interpretations and uses toward even the same notation (table or $y=mx$), as she became more comfortable with it. The quality of interpretation she developed was rather idiosyncratic. It reflected subtly the nature of her proportional reasoning. Nevertheless, it was the interpretation that served her making sense of novel problems during class as well as $y=mx$ as equation and that underlay her extended use of unit factor approach.

Theoretical Background

Proportional reasoning is a form of mathematical reasoning that involves a sense of covariation and of multiple comparison, and the ability to mentally store and process several pieces of information (Lesh, Post, & Behr, 1988). It is a shared understanding that proportional reasoning does not emerge as a full-blown ability, but that it develops gradually by increasing local competence. However, we do not know much about how children's proportional reasoning develops through their learning experiences under instruction in school. Research on proportional reasoning have identified developmental stages; but they were mainly from a larger sample in the laboratory setting. Recently, children's intuitive, context-bound, and often presymbolic solution strategies even before instruction have been documented (e.g., Hart, 1984; Lamon, 1993). Yet it has not been studied substantially about the consequence of these strategies during instruction, beyond a warning that children tend to memorize formal procedures such as cross-products mechanically.

The purpose of this study is to get information about such developmental process of proportional reasoning by examining in depth the learning processes that students go through while getting explanation and practice concerning proportion in the classroom. I used naturalistic method of inquiry with a small number of students at various levels of proportional reasoning ability. In this article, I concentrate on one of these students, i.e., a Japanese sixth-grade girl named Hitomi whose achievement was about average.

A characteristic of this study is to try to understand children's development of proportional reasoning through their real-time experience under instruction in ratio and proportion in the classroom. The reason for it comes from an evolving understanding that a person's learning is tied specifically to context in which it occurs, especially his/her interpretation of and interaction with the context (Rogoff & Lave, 1984). Indeed, there is ample evidence that contextual peculiarity of mathematics classroom influences children's learning of mathematical concepts (e.g., Saljo & Wyndhamn, 1990). When most studies on proportional reasoning used interviews or written tests in the laboratory setting, this study intends to offer an information of children's learning actually occurring in the classroom. Another characteristic is that it sheds light on children's interactions with notations and representations. There are both theoretical and empirical reasons for this. Theoretically, the key role that notations play in mathematical constructive processes is pointed out (e.g., Kaput, 1991). In ratio and proportion, specifically, different representations including table and algebraic expression refer to different experiences. It suggests that coming to know these representations promote children's reflections and connections. Empirically, while observing and analyzing students' learning processes in this study, I began to recognize that their reactions to newly introduced notations are rather visible and mirror important gains or losses in their proportional reasoning (Ito-Hino, 1995).

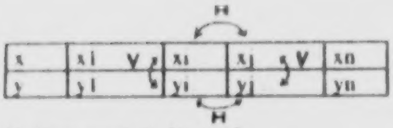
Teaching Proportion with Different Representations

In Japan, the idea of proportion is taught from earlier grades in elementary schools. For example, when teaching multiplication table in grade 2, teachers emphasize the relationship between multipliers and products as a rule of multiplication. In grade 5, "quantity per unit" (e.g., 60 kilos per hour) is introduced in order to compare two quantities: here, proportional relationship is also assumed to underlie the two quantities. In grade 6, these earlier experiences are reflected on and for the first time, proportion is defined and its mathematical characteristics are clarified along with different representations, i.e., table, graph, and algebraic expression $y = (\text{fixed number}) \times x$ (in this article, an abbreviation $y=mx$ is used).

Table below is a brief summary of content in each of the total 12 lessons in proportion that the teacher in this study organized.

Day(s)	Summary of Content
1	Finding examples of two variables changing dependently
2-3	Different relationships are expressed by using tables. Proportion is defined with the table (Day 3)
4	Exercises on choosing examples of proportional relationship. Making sure of existence of "fixed number" (y/x) in the proportion table
5	Summary of three ways of finding the "fixed number" from the table. Introduction of algebraic expression of proportion $y=mx$.
6-8	Exercises such as identifying proportional situations, filling in proportion tables, expressing the relationship in the form of $y=mx$, identifying the meaning of "m", or finding missing-values in proportion problems. Introduction of graphic representation (Day 8)
9	Strong points for each of the three representations of proportion
10	Solving problems by using different representations
11-12	Group exercises

Two observations of the teacher in treating the proportion table and $y=mx$ are noted. First, the teacher intended students to recognize connections between the table and $y=mx$, especially, the connection between the regularity in the table, i.e., uniquely determined quotient y/x for any corresponding values in x and y changing proportionally, and "m" in $y=mx$. In doing that, he initially made a distinction with respect to relationships between values in proportion table: "horizontal" and "vertical." The "horizontal" relationship concerned the multiplicative relationship among values within a row, while the "vertical" relationship is about the values between rows (see figure 1). By making such distinction, he led the students' attention to the



H Horizontal, V Vertical

Figure 1 "Horizontal" and "Vertical" relationships

latter. Once they recognized that there is a unique "vertical" relationship, he introduced the term "fixed number" (meaning y/x) that determines the relationship. The expression y=mx was introduced based on this understanding. Second, soon after the introduction of y=mx, the teacher came to emphasize its use as an equation. In almost every lessons, he explained the way of substituting a value for x (or y) in y=mx in order to get the corresponding value of y (or x). He tried to have students understand an efficiency of the algebraic expression in finding missing-values.

Method

My everyday visit to the classroom started in April 1993. In May and June, a pre-assessment of the students' proportional reasoning was conducted. Based on the results as well as their behaviors in class and the teacher's comments on them, I chose four target students. I began observing them in late June; it continued until the end of October. During the time, the teacher covered several textbook chapters; the chapter on proportion was dealt with in October.

Hitomi was 11 years and 7 months old when the teacher went into the chapter on proportion. She was chosen as one target student based on the pre-assessment as representing an average student in the range of achievement. Hitomi was also considered to be conscientious who worked hard in her class. She was cheerful and open to communicate with me, which was another reason for choosing her in order to get a thick description of her learning processes.

In the observation of Hitomi, I tried to identify the benefits she acquired from her work in the class and the learning processes she went through in acquiring that knowledge. In each session, I either videotaped or wrote down on paper her behavior throughout the session. I also audiotaped her voice by a small tape recorder that was regularly put on her desk and further, collected her notebooks and worksheets. After each session, I collated these data and developed a description of her behaviors. On a regular basis, I also interviewed her and asked to solve missing-value proportion problems as well as about the work done in class.

From the results on the pre-assessment and interviews regularly held, prior to instruction in proportion, Hitomi had been relying on the abbreviated build-up processes[1] when the problem involved easy ratio complexity, which places a relatively low demand on a rate conception (Kaput & West, 1994). Although she sometimes used the unit factor approach[2] that needs the rate conception, her use was strictly restricted to the shopping context in which thinking about unit price is rather natural. Overall, her use of strategy was context-bound: she did not have any general method that works for a large range of proportion problems.

Construction of Meaning of Table and y=mx During Instruction in Proportion

In this section, I illustrate the learning processes Hitomi went through during the class in proportion, especially, her view and use of notations introduced by the teacher and dealt with through various activities.

Two Ways of Reading Proportion Tables

During the first five lessons the students were engaged in activities with tables. As described earlier, the teacher distinguished two relationships between values in the proportion table. Hitomi naturally incorporated the "horizontal" relationship. For example, she developed a table (Figure 2) by saying "This is 10 minutes and 10 + 5, it's doubled, so this is also doubled and it's 50 cm. If I make this 100 cm. and 100 + 25 is 4?... it's 4, this is 4 times as much, so this is also 4 times as much... it would be 20 minutes..." She even showed this horizontally-oriented view in Day 1, when making examples of two quantities changing dependently.

In contrast, she did not easily recognize the "vertical" relationship. It was a boy who first referred to the relationship in response to the teacher's question about a table in Figure 3 (Day 3).

Time (min.) x	5	10	20	30
Depth (cm.) y	25	50	100	150

Figure 2 A table that expresses a proportional relationship between x and y

Time (hour)	1	2	3	4
Distance (km.)	4	8	12	16

Figure 3 A proportion table dealt with in class on Day 3

T We are looking at the table horizontally like this (pointing to the arrows)... Suppose there is a missing part, and we want to fill in that part, what other ways there would be, do you think?

S Vertically? I divided the numbers in the lower row by the corresponding numbers in the upper row, and then I got all 4s I've been wondering what that would be...

T Oh Do you understand what he said? He read the table vertically...

At this moment, Hitomi stared at the board. She appeared to check whether what the boy said was correct by dividing each number by its corresponding number. As far as my observation can tell, this was the first time that she made explicit the multiplicative relationship between x and y in the proportion table.

Derived Interpretations to the Proportion Table

I identified at least two ways of using the proportion table that Hitomi developed through her class work. They were either different from the method taught by the teacher, or not given explicit attention in the class. Due to the space limitation, I describe one of them which was fundamental in her process of learning.

A use of proportion table that Hitomi developed was to fill in the missing-value y_n in the table by using the value of y_1. Here, she paid special attention to y_1 and found the missing-value y_n via multiplication of y_1 and x_n (as for the names y_1, y_n, or x_n see Figure 1). This interpretation of the table was first observed in Day 5 when different ways of finding the "fixed number" was summarized. After the lesson, I interviewed her about a "fixed number" in a table (Figure 4).

Time x (hour)	1	2	3	4
Distance y (km.)	3	6	9	12

Figure 4 Part of a proportion table dealt with in class on Day 5

I You worked with this table today What is the fixed number in this?

S It's three!

I Why?

S Well if you do 3 + 1, 6 + 2, and so on, you can get 3 every time you do division

I Were there any other reasons in class? Or do you think there are any

S (giggled) I wonder if I remember What I remember is well since y is 3 since y is 3 if you double this (pointing at 2) you can get 6, and if you multiple like this way (pointing at 3 in y and at 3 in x) you can get 9."

However, what she remembered was slightly different from what the teacher explained in the class. He treated the "fixed number" more as a generalized relationship that satisfies all (x, y). This derived interpretation by Hitomi seemed to last for some time (until Day 10).

Initial View of Algebraic Expression of Proportion

Day 5 was also the day when the algebraic expression of proportion was introduced to the students. As described, the teacher introduced y=mx as a generalized relationship between x and y and emphasized its use as an equation to find missing-values. On the other hand, Hitomi was developing her own view toward that new expression.

In Day 5, she was observed not to put a "fixed number," which she found from a table, in the blank in y = (blank) × x until the teacher instructed the students to do so. After the class, I interviewed her about the same table in Figure 4 which was also dealt with in the class. Hitomi, after identifying several relationships in terms of "fixed number 3" in the table, was asked about the algebraic expression of that table.

I Would you write an expression using x and y about this table?
S X and y multiplication? Oh, I see, we did that today. I think we wrote something like this (writing "y = fixed number × x = ?") I think this is somewhere at the bottom of the textbook
I I would like you to think about this table. What would that be in this case?
S (Writing "3 × 1 = 3, 3 × 2 = 6, 3 × 3 = 9, 3 × 4 = 12" vertically)
I How about using letters x and y?
S (Looking at the algebraic expression she just wrote) y equals fixed number multiplied by x
I Let's see how will that be in this table?
S (Pointing at the four numerical expressions she wrote) That will be like this!

The protocol indicates that she, in spite of her recognition of several relationships between values in the table, did not see connection between them and the algebraic expression. Once she was asked to write an algebraic expression, she forgot all about those relationships and tried to recall a specific symbolic notation that was written on her textbook. The protocol and other observations of her writing of algebraic expression also show that she used "y=mx" as a seal of proportional relationship. Indeed, when asked to translate the table in terms of x and y, she wrote "y=mx" literally, instead of writing "y=3x." Initially she wrote "y=mx" and "y ÷ x" (or "x × y") interchangeably, or added odd symbols as shown in the protocol above. Hitomi soon came to add "(5)," for example, beneath "m" in "y=mx," instead of writing "y=5x." (Day 6); still, the standard writing of algebraic expression had not observed until later around Day 10.

Emergent Use of y=mx as an Equation

In spite of her teacher's repeated explanations of y=mx as an equation to find missing-values, Hitomi had not been using y=mx for that purpose. When she needed to find missing-values, she searched them directly from problems without formulating the algebraic expression, or made use of tables or graphs when they were available. It was around Day 10 when she began to use y=mx in order to find the missing-value.

In Day 10 exercises, the students were finding the distance that a car at a constant speed of 40 km/h would take to drive in 7.5 hours. Hitomi first answered the question by computing 7.5×40. Here, she did not intend to formulate y=40x and substitute 7.5 for x. Indeed, when the teacher explained the use as equation, she whispered "oh... yeah it works." In retrospect, this was her turning point. After this, she solved another problem by first formulating y=3x and then substituting 10 for x. In the interview session conducted after the class, I gave her another missing-value speed problem. Here again, she used y=30x developed in a previous question to substitute 7 for x in order to find the distance that the car goes in 7 hours.

It is not clear why her use of y=mx as an equation to find missing-values emerged. Probably one of the reasons would be that she had listened to the teacher's explanation repeatedly. However, Hitomi did not just memorize the repeated explanation of the use of y=mx as equation. In an interview, when asked to write an algebraic expression from a table she developed for a proportion problem, she added two values 1 and 35 outside (Figure 5) and then wrote an expression "y=35x." Being asked the reason for 35, she pointed at 3 in the table and said, "Since this is 3, I made it 1... so it's 35." This observation implies that she made a correspondence between "m" in y=mx and "y_1" in the table. In other words, she acquired the new use of y=mx by making connections with her own interpretation of proportion tables that she had developed earlier. Since then, she came to use y=mx not just as a seal of proportional relationships but also as a tool for finding missing-values. She also came to call y=mx as "formula."

1	Time (hour)	3	6
35	Distance (km)	105	210

Figure 5 A table and Hitomi's addition to it

Discussion

Hitomi, like other students, brought her conception of proportion into instruction under which she encountered notations of the proportion table and y=mx. Her learning process documented above shows that she had been developing her own interpretations and uses of these notations from the very beginning of her interactions with them. They were also changing as she became more familiar and comfortable with them.

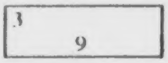
The quality of interpretation that she developed was not identical to what the teacher intended. It was affected subtly by Hitomi's view of proportional situations that was predominantly build-up-based. In the case of proportion table, she began to incorporate her build-up-based view into the "horizontal" relationships between values in the table. She then derived the special attention to y_1 and the way to find missing-value y_n via multiplication of y_1 and x_n, whereas the teacher treated the fixed number more as a generalized relationship that satisfies all (x, y). Here, she seems to have avoided (perhaps unintentionally) to directly grasp y/x as the general multiplicative relationship, which will require her weak rate conception. Interestingly, the attention to y_1 seemed prevalent among the students. Three of the four target students showed their focus on y_1 in the table. In class, different wordings of y_1 such as "base number," or "the first fixed number" were also heard from other students.

Although they were more subjective, these interpretations came to play a crucial role in organizing her thinking processes as "generative mental operations" (Kaput, 1991, p.55). Indeed, Hitomi came to be able to identify proportional situations in terms of "fixed number," fill in proportion tables, and solve missing-value problems by using tables. The derived interpretation with the emphasis on y_1 also served her in treating y=mx as equation. It further enabled her different ways of solving the missing-value problems, as described in the last paragraph.

Concerning y=mx, for a long time she viewed it as an authorized seal of proportional relationship rather than a thinking tool (Ito-Hino, 1995). It was near the end of lessons that she began to use it as an equation to find missing-values. This rather monotonous interaction, compared to that with tables, would reflect the extent to which a notation is prestructured. The expression y=mx was highly prestructured that demanded her use of rate conception, whereas the table was more open in which her build-up-based view was supported. For her y=mx would have been difficult and foreign in comparison with the table. Under such circumstances, the emergent use of y=mx as equation is to be considered as a sound progress to her. Note that it became possible by her derived interpretation of table. Here again, she incorporated her build-up-based view into the new use of y=mx. She interpreted "m" as the specific value y_1 to enable a multiplication of the number of known-quantity, which is 1 in this case, increments (x) by the size of the unknown-quantity increment $(y_1)^{1)}$.

The observations of Hitomi's proportional reasoning strategies suggest that her derived interpretation of proportion table further made a contribution to her repertoire of strategies. Before instruction, she had not used unit factor approach except for shopping problem context. Instead, she had been relying on abbreviated build-up processes. For example, she solved a rectangle problem below by using the processes, i.e., 564 ÷ 3 = 188, so 188 × 9 = 1692. After instruction, she solved the same problem quite differently:

Problem *(With an original 3-by-9 rectangle written on grid paper.) suppose you want to draw a very big rectangle If you make the length of the vertical side 5 meters 64 centimeters, how long should the other side be?*



S 564 × 3 = 1692? centimeters I think this would be okay
I What do you mean by 3 here?
S Well, this rectangle I changed it like this (drawing a 1-by-3 rectangle on the sheet) I'm not sure about this, but this is the first size, and then I would multiply by 3 here (pointing at the width 3 of the 1-by-3 rectangle) this width well this is the base one base number? well it would probably be the fixed number 18 ÷ 6 (meaning the 6-by-18 rectangle that she drew before this question) is also the same number I mean multiply the fixed number by this 564, it would be 1692 centimeters

The protocol shows that Hitomi created a unit factor (3cm per 1cm) and multiplied it and the given total quantity 564cm to determine the total amount of the unknown quantity. Actually, this was her first use of unit factor approach in problem contexts other than shopping. It further shows clearly that her derived interpretation of proportion table underlay her unit factor approach. She found the missing-value x in 564-by-x rectangle by identifying the fixed number y_1 (3) in an imagined proportion table and multiplied the y_1 by the 564. The importance of unit factor approach in bridging between children's intuitive strategies and formal method that fully uses the rate conception has been pointed out (e.g., Herron & Wheatley, 1978). The result of this study suggests an unexplored complexity of the bridging role of unit factor approach. Further investigation is needed how children acquire extended use of unit factor approach from essentially intuitive one and in what way learning of notations contributes to the process.

Concluding Remarks

In this article, it was illustrated that a student's learning of different notations of tables and y=mx during the class work was deeply interwoven with her proportional reasoning. Her interpretations and uses of even the same notation were changing with her increasing familiarity with it. The interpretations being developed were rather idiosyncratic, subtly reflecting the nature of her proportional reasoning. Nevertheless, it was these interpretations that she based in making sense of novel problems during the class and y=mx as equation; and it was those that underlay her extensive use of unit factor approach.

Further analysis needs to be made concerning both the nature of derived interpretations of notation and its relationship to proportional reasoning. In doing that, consideration should be given to not only purely cognitive aspect but also other aspects. In the case of Hitomi, she was more salient in her disposition toward following what the teacher told and shown, compared with the other target students (Ito-Hino, 1994). For example, she was the only student among the four who spontaneously used the tabular notations being dealt with in class in solving proportion problems in interview sessions. Such interests in the teacher-taught methods and notations and their dispositions toward learning mathematics in class should also affect the learning processes under instruction.

Note 1) The abbreviated build-up process consists of "a quotitive division that determines the number of known quantity increments, followed by a multiplication of that number of increments by the size of the unknown quantity increment" (Kaput & West, 1994, p 249).

Note 2) The unit factor approach consists of first obtaining the unit factor by dividing a quantity by the other in the problem and then multiplying the factor by the given total quantity.

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COMMUNICATING TEACHER'S METAKNOWLEDGE THROUGH LESSONS

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How can we help teachers to communicate their metaknowledge, especially, the nature or characters of mathematical knowledge or activity to the students? We regard this as the central research question of our study. The present paper can be understood as an attempt to try to ask the question through consideration of practices in the classroom. To do so, a cooperative teaching experiment was conducted over a four-week period including 4 lessons with 8th grader's class on June 9, 12, 13, 16, 1995. The analysis of interaction between the teacher and students in the lessons revealed that when (1) the teacher helps students to develop their thinking tools which they already have used to approach the task concerned, and further (2) the developing their thinking tools enable students to aware or discover what they have never known, the teacher's metaknowledge can be communicated to students.

BACKGROUND

According to Schubring(1991) the concept of metaknowledge is derived from american discussion related to curriculum reform in the 1960s. The discussion revealed the significance of teacher's view of knowledge as well as his/her knowledge. Smith(1969) has diffused this idea with a distinctive term 'knowledge about knowledge'. Further, the idea of "knowledge about knowledge" have been elaborated theoretically from an logical-epistemological point of view by the researchers of IDM in Germany. They have introduced the term "metaknowledge" in mathematics education and have indicated its significance in relation to teacher education(IDM, 1981; Keitel & Otte, 1979; Seeger & Steinbring, 1986). Schubring(1978) has pointed out the new significance of history of mathematics as metaknowledge. He seems to suggest history of mathematics is one of the view points which helps teachers to develop their metaknowledge. Iwasaki(1992; 1995) have developed some examples which help teachers to be aware of the significance of metaknowledge.

Recent studies have investigated relationships between teacher's conceptions or view of mathematics and his/her teaching practices by the case study(e.g., Thompson, 1984). Further, Lerman(1983) has proposed problem-solving teaching perspective can initiate substantial changes and advances for school mathematics programmes. These studies seem to suggest that it is important for teachers to develop their metaknowledge more and more. However, if teachers develop their metaknowledge, their students can develop their metaknowledge which compatible with teachers one? Cooney(1985) have revealed the conflicts between a beginning teacher's idealism and the reality of classroom practice.

How can we help teachers to communicate their metaknowledge, especially, the nature or characters of mathematical knowledge or activity to the students? It seems this problem still remains to be solved. Thompson(1992) points out,

> "we know little about how instructional practices, in turn, communicate those conceptions or others to students, if they do so at all. Insofar as one observes congruence between the mathematical beliefs of students (Schoenfeld, 1983) and those of teachers (Thompson, 1988), it is natural to infer that some communication is effected. Furthermore, since teachers are the primary mediators between the subject matter of mathematics and the students, it is also natural to infer that the teachers' conceptions are indeed communicated to students through practices in the classroom. This chain of inferences, however, remains to be empirically validated."(Thompson, p.141)

How can we help teachers to communicate their metaknowledge[1], especially, the nature of mathematical knowledge or activity? We regard this as the central research question of our study. Some researches seem to suggest that it is significant for this problem to develop social norms or morals in the classroom. (Cobb & Yackel & Wood, 1990, 1992; Lampert, 1990; Kumagai, 1995) The present paper can be understood as an attempt to try to ask the question through an analysis, from an epistemological point of view rather than a social point of view, of interaction between a teacher and students in the classroom.

METHODOLOGY

Cooperative Teaching Experiment

The author conducted cooperative teaching experiment with the teacher over a four-week period. The teacher teaches and studies in a junior high school attached to Joetsu University of Education with a subject of study: ' Developing teaching materials to foster a creative view of mathematics'. This is the reason why he has been selected in this study. He is an experienced teacher who has a master's degree.

The procedure of lessons plan was as follows: The author and the teacher met several times. The author made and explained his concrete teaching plan of the lessons. The teacher mainly criticized and modified it from a practical point of view. In this way, we elaborated the teaching plan of lessons cooperatively. But the plan was not fixed, rather, in order to achieve our common aim it have always been able to stand further improvement in the process of teaching experiment.

The method here is compatible with the idea of 'kritisch-konstruktiver Erziehungswissenschaft' in Germany(Klafki, 1976). Conducted cooperative teaching experiment, metaknowledge which is hardly explained would have to be explicit.(see the section 'Knowledge and Metaknowledge')

Data

The data were collected in the course of the cooperative teaching experiment. Four lessons were conducted with 8th grader's class on June 9, 12, 13, 16, 1995. The subject matter of the lessons were the introduction of triangle congruence theorems. All of lessons were recorded with two video cameras in front and behind the classroom. The front camera was used to record the whole-class discussion and student's behavior and the behind camera was used to record teacher's behavior respectively. Transcripts of the lessons were made on the bases of these data. Further, some retranscriptions were made from the following point of view:

> When making the actual analysis one cannot simply use such a step-by-step procedure, ···one has to analyze carefully the epistemological status of the mathematical knowledge from phase to phase ···to explore the development and shifts of knowledge interpretation and understanding in the classroom interaction. The method of analysis is a dialectical one reflecting global and local aspects simultaneously.(Steinbring, 1993, p.38)

Knowledge and Metaknowledge

We assume that there is no knowledge without metaknowledge(Bateson, 1973; Otte & Bromme, 1978; Mellin-Olsen, 1987). We distinguish between knowledge and metaknowledge in this paper as follows: Knowledge is related to the relation between things, on the other hand, metaknowledge is related to the relation between thatrelation and the subject (Otte & Seeger, 1994). That is, the former is related to triangle congruence theorem as a relation, and the latter is related to what the triangle congruence theorem is all about

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[1] We, as IDM researchers, prefer to use the term 'metaknowledge' rather than 'conception' because the term metaknowledge seem to express important cognitive functions of hierarchal relationships with knowledge better.

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for us. We shall make our metaknowledge clear here. Following view of triangle congruence theorem is based on some historical and epistemological studies of geometry texts (e.g., Nakamura, 1981; Hirabayashi,1991; Reed,1995). 'What is the triangle congruence theorem' should include (1)an activity trying to grasp the 'putting two triangles on top of each other perfectly' by elements of triangle, sides and angles more clearly, that is, "logische Analyse unserer räumlichen Anschauung"(Hilbert,1930) (2)thinking tools to recognize geometric relations, and (3)Greek ways of thinking trying to grasp the whole with some minimal elements, that is grounds for geometric system. In short, our metaknowledge here can be characterized by 'Hilbert's view of geometry' and 'pragmatism'.

To Evaluate Student's Metaknowledge

Metaknowledge as mentioned above is related to how to define the triangle congruence theorem. Accordingly, to evaluate student's metaknowledge we have to ask to students what the triangle congruence theorem is or what it should come into being. It is difficult for them to ask these questions directly.

In teaching experiment we provided the situation in which some students are asked whether they accept the SsA (On the notation, see Hirschhorn, 1990) theorem or not. If students can accept or reject new theorem for them without dependence on teacher, we can infer that students, to a certain extent, understand what the triangle congruence theorem is or what it should come into being. In Japan, there are not SsA theorem in the textbooks now in use. The theorem is new for students. Further, asking students their criterions, we can understand their metaknowledge.

ANALYSIS AND DISCUSSION

The Lessons

The lessons consisted of two main phases, Phase I and Phase II. In Phase students were able to use their intuitive ideas of triangle congruence theorems in order to justify some geometrical relationships but they were not able to say why the two triangles congruent to each other. So the teacher tried to make students notice that they intuitively used the idea of congruence and had to use it more logically. In Phase II to concrete shape this problem the teacher formulated it as the present task:"*There are two triangles* ABC *and* $A'B'C'$ *where* $AB = 5.6cm$, $BC = 5cm$, $CA = 4.1cm$, $\angle A = 60°$, $\angle B = 45°$, $\angle C = 75°$ *and* $A'B'C'$ *are unknown. What conditions must be satisfied when they are congruent to each other?*"

First of all, the teacher asked "In the case of one condition, for example, $AB = A'B' = 5.6cm$, these triangles are congruent to each other?" All of students said "No!" He asked "Why?". One of the students explained "There are many $A'B'C'$s satisfy it. The point A'can move anywhere!". Then he asked "In the case of two conditions, for example, $AB = A'B' = 5.6cm, BC = B'C' = 5cm$, these triangles are congruent to each other?" All of students said "Of course, No!" He asked again "Why?". One of the students explained "There are many $A'B'C'$s satisfy them. The point A'can move on the circle with the radius 5.6cm centered at B'." In this way the teacher initiated students into construction as *thinking tools* to confirm whether the set of conditions which they found is congruence theorem or not. Furthermore, he asked "What do you want for $A'B'C'$or a point A'?" Some of the students explained "We must have only one $A'B'C'$or point A'." In this way it was shared in the classrooms that "if you can construct the only one triangle when you constructed under a certain condition then the condition have to be congruence theorem".

The teacher suggested to students that they should enumerate conditions one by one on the number of conditions. In the case of one condition, then, there are 6 sets of conditions. It was trivial for students that all of these can not be congruence theorem. In the case of two conditions, there are 15 sets of conditions. It was also trivial for students.

Emerging the Contradiction

Students was beginning to infer what the teacher's intention is. If he had been taught as usual, their inferred-intention would be appropriate. But it was inappropriate. Because the

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teacher have changed his teaching style in the process of planning sessions of the lessons for the teaching experiment. His original intention of the lessons was to present the triangle congruence theorems in the textbook now in use and to ascertain the adequacy of them by construction. In fact he said in a planning session of the lessons "I often finish the lessons concerned in five minutes." or "How can we help students to discover what they have already known." It means, for students, saying what they have already known about congruence theorems could be a solution of the present task.

Consequently, the teacher's change of his teaching style have required students to understand that what they regard as a solution is not the solution in the lessons. The following interaction makes this contradiction clear.

Situation 1 (I,p156-164)

156 **T:** OK? for example, well then, if you have to make sure that these two triangles are perfectly congruent to each other, what conditions must be satisfied? Kubo please.
157 **kubo:** Well, all sides (of the triangle ABC) are equivalent to (side of the triangle $A'B'C'$)respectively.
158 **T:** Equivalent? OK, All sides (of the triangle ABC)are equivalent to. And? Kusa what? All sides and?
159 **Kusa:** well, the length of one of sides somewhere ···
160 **T:** Stop, stop! Wait for a moment. In order to make sure, all sides and?
161 **Kusa:** (He is looking at his friend behind him putting his head a little to his side.)$A'B'$, $B'C'$ ··· All sides.
162 **Kusa:** eh!
163 **T:** Now, all sides, when this(triangle ABC) and this(triangle $A'B'C'$) are exactly equivalent, all sides and? you have to know what?
164 **Kusa:** That's all.

Obviously, the teacher expected students to state the definition of triangle congruence, however, students tried to state determinating conditions that they have learned at primary school or triangle congruence theorem they have learned at a private school. They seemed to perceive that the task was aimed at confirming what they have already known: determinating conditions. So, for students, saying them was solutions of the task. If the teacher had been taught as usual, it would be appropriate. But it was inappropriate. Protocol 161-164 shows this contradiction clearly. This implies implicitly student's *metaknowledge* which they would have developed through his teaching so far. That is, mathematics (triangle congruence theorems) consists of a set of well-known facts (SSS, SAS, ASA), consequently mathematical activity (inquiring triangle congruence) is to confirm these facts.

Surprising Fact for students

In the case of three conditions, there are four cases: three sides (SSS), three angles (AAA), two sides and one angle (SSA), one side and two angles (SAA). First of all, the set of conditions "three sides (SSS)" was investigated. From the look of their expressions, we may safely conjecture that students didn't need to investigate the case, but the teacher offered to investigate the case. Students accepted it somewhat unwillingly. The teacher constructed the figure which satisfies the conditions($A'B' = 5.6cm$, $B'C' = 5cm$, $A'C' = 4.1cm$) on a blackboard. Then students help his construction with their voices. Note that there are teacher's positive intervention here.

Students naturally accepted the figure which had two intersection points of the two circles(see, Fig.1). However, It was surprising fact for students there were, as the very natural result of the construction, two A'above and below the side $B'C'$respectively. That is, They have seen only one A'above the side $B'C'$.

Situation 2 (II,p249-262)

249 **T:** How many points do you have?
250 **S:** One! One!
251 **T:** eh?
252 **S:** (in haste)Oh! Two!
253 **T:** One?

254 **S**: Oh! Two! Two!
255 **T**: What's this?(pointing at a point above)
256 **S**: A'.
257 **T**: A'. What's this?(pointing at a point below)
258 **S**: A'.
259 **T**: We have constructed it based on the conditions. (encircling the condition written on the blackboard with his yellow chalk)
260 **S**: Yes.
261 **S**: Oh! I see!
262 **T**: Including the point(below), we have two points!

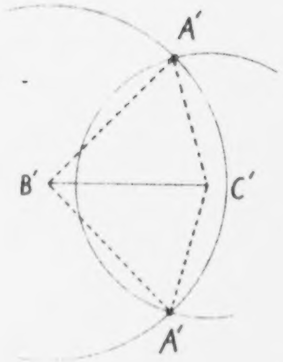

Fig.1

Situation 3 (II, p263–270)
263 **T**: Oh?!, in that case, Is this OK? (drawing the side $A'B'$and $A'C'$above)
264 **S**: A'haven't been fixed!
265 **S**: It seems to be right!
266 **T**: So, here, we may include ⋯ (drawing the side $A'B'$and $A'C'$below), Is this OK?
267 **Yama**: Oh! I see what you mean!
268 **Kusa**: What a wonder!
269 **T**: eh! What a wonder? Do you have only one (triangle)? Only one?
270 **S**: Don't!

Situation 3 shows that students have not seen the other possible triangle $A'B'C'$below the side $B'C'$although they have recognized two points which satisfy the conditions. Namely, they have tried to construct the same triangle as the triangle ABC. With this as a turning point, students negotiated how to confirm whether the given conditions are congruence theorem or not. As a result of this, they changed their way of confirming from "if you can construct the *only one triangle* when you construct under a given set of conditions, the set of conditions have to be triangle congruence theorem" to "if you can have *only one sort of triangles* when you construct all triangles which satisfy a given set of conditions, the set of conditions have to be congruence theorem".

This suggests that their awareness of "constructing all triangles which satisfies a given set of conditions" rather than "constructing the same triangle as the triangle ABC,which satisfies a given set of conditions" is gradually growing. Thus their way of inquiring with construction which have been initiated by the teacher have become their thinking tools in order to investigate congruence theorems. Finally, developing their thinking tools enable them to discover a new relation described next section.

A Student's Discovery

There was a student's discovery when in the case: two sides and one angle (There are 9 sets of conditions, see Fig.2) students investigated the possibility of conditions individually.

Situation 4 (II,p440–463)
440 **T**: Prolong your investigation!
441 **Yama**: Look!
442 **Yama**: I say, try the next of the center.
443 **Yama**: Something wrong! Oh no!
444 **Yama**: Look!
460 **T**: Then, I'm going to explain how to construct ⋯ If it is difficult for you to construct the figure which satisfies the conditions, you had better draw the side which have the angle first. If you do so ⋯
461 **Yama**: There is no difficulty to do so, teacher!
462 **S**: There are two solutions.

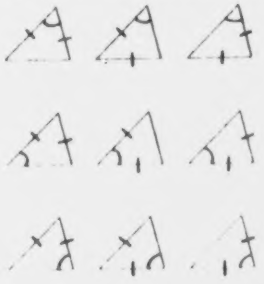
Fig.2

463 **Yama:** It goes beyond the level whether it is difficult or not!

As the student Yama said in the class a few days later " I have happened to draw a little large circle at that time." (IV, p22), the discovery is derived from experiential facts. In other words, if he had not drawn the figure where the circle met the side at two points, he could not discover it. On the other hand, It is hardly considered that the discovery is only derived from the experiential fact. He would have to be able to see each of the two points equally as the point which satisfies the conditions. Because , at that time, if he had been conscious of constructing the same triangle as the triangle *ABC*storongly, he could have ignored an intersection point below which satisfies the conditions.

In short, It is necessary for his discovery that there is experiential fact and moreover he can recognize intersection points, as a result of the fact, which satisfies the conditions. It is notable that developing his thinking tool initiated by the teacher enable him to discover.

We want to stress that, with this discovery as a turning point, there have been a change in the classroom atmospherethat is, we haven't seen the interaction as situation 1 ever after. Such interaction occurs when a student says his/her congruence theorem which is not appropriate from the context. Following interaction shows this change.

Situation 5 (IV,p142–147)

142 **T:** After that, we had such conditions(SSA) left over. In such conditions, which sets of conditions have established only one sort of triangles? What? You want to say something? Ok, please.

143 **Naka:** Yes. ···let me see, two sides and the angle between them, ···one side and ···let me see, two angles which put it between them. These are conditions of ···

144 **T:** eh? OK. Wait for a moment! It is the next, isn't it? Now, we try to do the case two sides and one angle. OK. OK. Thank you. We will have to investigate what Naka said. What? Yama please.

145 **Yama:** Yes, first of all, 2 sets of conditions below and a set of conditions right-above are established.

146 **T:** Yes.

147 **Yama:** And, what I ought to do is to tell why (they have been established), isn't it?

In this situation, Naka said congruence theorem ASA which she have already known. However, there was no statements with the conflict of students such as "That's too bad."(I, p355) or "You should not say such a thing!"(I, p363). That is, There are no interaction described above. This shows that students have begun to understand their solutions or answers is not the solutions or answers here.

Grounds of Students

Yama explained why six SSA conditions were applicable in some cases and were not in the other cases. Advices of some students sometimes helped him. Students referred to that the applicability depended on sizes of two sides in the set of conditions. Then the teacher encouraged students to formulate the set of conditions. They formulated it as follows: S-S-A and "the side with an angle ≤ the side with a side ". After the teacher sorted out the triangle congruence theorems that had been shared in the classroom, he asked whether the set of conditions above could be a triangle congruence theorem or not. Students answered they could not be. Their grounds were as follows:

> **Yoshi:** Because if we could established it on most few conditions, this would be the best. Three sets of conditions above have only three conditions, so these are OK. But this set of conditions is not the minimum, for it has S-S-A and further one condition. (IV, p347-355)
>
> **Yama:** This set of conditions fussy about given triangles. The triangles must be satisfied it. But if we applied the set of conditions S-S-A two given triangles then the additional condition would be not necessarily satisfied. You will want to state we could use it every triangles. But it seems to me that it fussy about given triangles. ···It is cheeky. So I don't want to state we could use it every triangles. (IV, p374-375)

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In their grounds, they have strong criterions. Yoshi's criterion seems to be the simplicity. In the other hand, Yama's criterion seems to be the applicability. Yama would not be able to express his thought appropriately, but he would want to state the additional condition could limit its applications. Their decisions, whether a set of congruence is a triangle congruence theorem or not, don't depend on the teacher yet. It means that the triangle congruence theorem is what they should create, especially on the bases of pragmatical criterions here, rather than what should be in the textbooks previously. It suggests they certainly begin to understand partly what the triangle congruence theorems is or what the activity inquiring the triangle congruence theorem means.

Their criterions seems to derive from the context of activities of the lessons. We explained somewhere they accepted SsA conditions, but their negative decision here seems to be consistent in the context here. If we will give students a situation where SsA conditions are very useful and ask the same question described above, they will be able to think about what the triangle congruence theorem is or what the activity inquiring the triangle congruence theorem means further and result of this they will understand it better.

CONCLUSION

The teacher have changed his teaching style in the process of planning sessions of the lessons for the teaching experiment. This change requires students to understand what they regard as a solution is not the solution in the lessons. The teacher helps students to develop their thinking tools which they already have used approaching the task concerned rather than present the triangle congruence theorems in the textbook now in use ascertaining the adequacy of them by construction. The teacher was trying to communicate what they regard as a solution is not the solution in the lessons, but in vain. As a result of this, It have caused the same pattern of interaction descrived above. But, a surprising fact for students and further a student's discovery as a turning point, there have been no this pattern of interaction. It suggests that students have become aware of the significance of inquiring the task with their tools and to understand what they should do at the same time. That is, students have begun to understand teacher's metaknowledge. Now, their solutions of the task is not to say what they already known but to inquire the task with their tools or their ways of thinking. Situation 5 and student's grounds described above show this.

The analysis revealed that when (1) the teacher helps students to develop their thinking tools which they already have used to approach the task concerned, and further (2) the developing their thinking tools enable students to be aware of or discover what they have never known and never would have known without it, the teacher's metaknowledge can be communicated to students better.

This conclusion illustrates the significance of *the individual's discovery of a thinking-tool* (Mellin-Olsen, 1987) in the classroom situation. Further, It suggests the importance of the teacher's positive intervention to the subject matter, since *the discover* is derived from his positive intervention descrived in the above section.

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A STUDY OF TEACHER ENQUIRY INTO THE PROCESSES OF MATHEMATICS TEACHING

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This paper describes a study which explores mathematics teacher research and its effects on the development of mathematics teaching. Issues emerging include the development of the research process by the teachers, the importance of collaborative structures and the centrality of mathematics to the study of teaching and its development.

Evidence has shown that when researchers explore the work of teachers through classroom observation and interviewing, the teachers' thinking and practice develop (Elbaz (1987), Jaworski (1994)). The questions asked by researchers distance teachers from their immediate concerns embedded in practice, cause reflection on events which have occurred and lead to a deep questioning of the teachers' underlying beliefs and theories. Such questioning results in a problematising of the teaching practice as teachers consider alternative approaches or seek to fulfil new objectives. The question this raises for research is: what force, other than an external researcher, might promote such deep reflection, problematisation and ultimate development of teaching practice in mathematics teaching?

One approach that is becoming familiar in mathematics education is that of teachers researching their own practice. This itself seems to beg many questions. How do teachers engage in such research? What is the nature of the research? What are its outcomes? What issues does it raise for the teachers who engage in it?

Background

Developing from the work of Stenhouse in the nineteen seventies, and from thinking of Lewin much earlier, the teacher-research movement has gained considerable momentum in recent years. Research into teaching and learning has come to mean research in classrooms, and teachers are central to classrooms. From research which has looked into the roles and acts of teachers from the outside, perhaps through valuing the contributions the teachers have made to the research the movement has developed a legitimacy for research conducted by the teachers themselves. There are many examples of research projects small and large where the chief researchers are teachers.

Such projects fall largely into two camps: larger scale projects in which a number of researchers work towards common goals, such as the Ford Research project in the UK (Elliott, 1991); and smaller projects conducted by individual teachers as part of some higher degree programme in which the research is a required part of the programme's assessment. An example of the latter, taking place at the University of York, was monitored for its effects on the practice of its participants (Vulliamy & Webb, 1992). Evaluators rated the success of the programme in terms of its contribution to the professional development of its participants more highly than its findings in terms of research results, although these were not insignificant. In many such examples, university researchers engage with the teachers in some form of tuition/ course leadership/ evaluation in which the outcomes of the research and its effects on the teacher-researchers are a source of study (e.g. Irwin, 1993). Collaborative practices are fostered in which university teachers and researchers and classroom teacher-researchers build mutually supportive relationships to further teaching development (e.g. Krainer, 1993).

As a result of such programmes it is evident that the practice of teaching benefits from the research undertaken by the teachers. The act of enquiring into aspects of ones own teaching might be seen to lead to a development of awareness which results in a greater sensitivity towards learning processes and the development of learners' knowledge and understanding. However, teachers themselves are often unfamiliar with research practice and traditional methodologies. They have little time to gain specialised knowledge about research methods, or to read widely in the literature relating to their area of study. Their questioning and enquiry might fail to draw on other relevant work and their approaches may lack the rigour of more formal research. If the research outcomes are not of importance, research rigour perhaps needs not to be a main focus of the research, which begs many questions about what aspects of the process might be considered to be research.

A collaborative project to explore mathematics teaching development

The Mathematics Teacher Enquiry (MTE) project, at the University of Oxford, was designed to explore the potential for the development of teaching of teachers undertaking research or enquiry into aspects of their own practice. It has no particular focus in determining knowledge into aspects of teaching or learning, and it leads to no form of certification. Its purpose is to explore the processes, outcomes and issues arising from teachers undertaking research or enquiry into aspects of their own practice and/or their students' learning of mathematics. A pilot study has been undertaken and a further study, drawing on its results, is currently being conceptualised.

A number of teachers volunteered to participate in the project, agreeing to undertake some form of research or enquiry. It was made clear in soliciting participation that the substance of their enquiry should be of direct concern to themselves, as would be their approaches to it. However, the university researchers (there were two) undertook to provide assistance with research methods or supportive literature as and when this became necessary. Collaboration between teacher researchers and university researchers in a reflexive relationship was seen to be a major objective of the research. Project meetings (two per teaching term) were an important feature of the research design, encouraging collaboration between all participants in the sharing of ideas and concerns, the identification of common issues and understandings, and a mutually supportive environment to sustain research. They provided highly significant data.

Major issues arising from this study include:

- The tentativity with which teachers first approached classroom research and its effect on the processes and practices which emerged.
- The development, for the teachers, of a meta-language in which to talk about theory and practice in the processes of mathematics teaching and learning.
- The role of *mathematics* in the study: is this a study of *teaching* where mathematics is 'merely' the focus of the teaching, or is mathematics itself a central feature of the learning, development and issues arising from the research?
- What is the nature of the teachers' research? In what ways does it fit established paradigms of research practice? Do we need to redefine research in these contexts? How does such research differ from good reflective practice?

This paper will focus on the last of these issues, while touching briefly on the others. These are discussed in Jaworski & Lee, 1994 and Jaworski, 1995. A report of the project is in preparation.

What is research?

Stenhouse defined research briefly and succinctly as "systematic enquiry made public" (Stenhouse, 1984). At first glance, the research undertaken by the teachers might be seen as being very *un*systematic. However, terms like 'systematic' derive their meaning from culturally and contextually related events. In its most positivistic sense, systematic can mean logically structured and predetermined. None of the research could be described in this fashion. However, in retrospect, each research had its own system, related to the thinking of the teacher(s), the research questions, however tentative, and its particular school environment.

In much (action) research undertaken by teachers, teachers are encouraged or required to keep journals in which they document their thoughts feeling and issues arising from their research. No such imposition was made in this project. The issue of writing, and its advantages was aired by the university researcher in discussion with a teacher and at project meetings, but no requirement to write was imposed. The majority of the teachers were reluctant to write, partly because of other pressures and lack of time, but importantly for the project, because, during the early stages at least, they saw no need to write, no purpose in writing, and basically they did not know *what* to write. Thus the making public the research was not a part of their thinking. Indeed, it was the case that they did not value their thinking or have confidence in their research, so that thoughts of making it public were almost laughable in the early days.

John Elliott (1991) defines action research as, "the study of a social situation with a view to improving the action within it". This seems much closer to the practice and processes in which the teachers engaged. Each one of them identified an area of interest or concern: there was something they wanted to find out, to know more about or to improve; often all three. Their enquiry was motivated by a desire to effect change in their classroom practice, or in their own knowledge and thinking related to that practice. The desired outcomes of their enquiry were not always clear at the start of the enquiry, nor were the methods which would need to be employed. The process of enquiry included clarification in both of these areas.

The nature of the teachers' research

During the project we used the word 'enquiry' largely instead of 'research'. It seemed more friendly, less forbidding word. *Research* has too many academic connotations, and some of he teachers were clearly inhibited by it. *Enquiry* captures well, not only *what* the teachers did, but also their developing thinking in doing it. This will be illustrated through glimpses of one teacher's exploration.

Sam had his own very clear objectives for the ways in which he wanted his students to work in mathematics lessons. When they worked in these ways he referred to their work as 'productive'; in other cases he talked of the students 'resisting' his preferred approaches to their learning. He wanted to know what happened in groups in his classroom when he was not present with them himself. How did students work together? What quality of thinking, discussion and negotiation of ideas was evident? However, he was not able to monitor the talk and interaction when he was not with a group; he could not be with all groups all of the time. Moreover, it was an important feature of his approach that they should develop independence and responsibility for their learning. His continued presence and direction would have countered these objectives.

In the beginning, after articulating the basis of his research, he sought help from colleagues who were willing to observe groups in his class. Whenever I visited him he used me for this purpose too. After a lesson, we would discuss the lessons and what I had observed, and Sam would relate this to his objectives for the lesson. Some of my observations proved salutary to him, when actions or thinking of the students were unexpected or ran contrary to his expectations. As a result of a number of such experiences he decided to develop his research in two ways: firstly to place a tape recorder with certain groups to record their conversation during a lesson, and secondly to interview some of his students to find out more about their views on their mathematics lessons and on his teaching. I did some of the interviews and he did others himself. He indicated that it had taken some courage to enquire into students' views of his teaching, but that the results had proved valuable for his learning. In particular, listening to the tapes, he discovered that he talked too much, and resolved to give more attention to listening to the students.

On one occasion, in his class, he asked students to draw a number line and place certain numbers on it. One group of girls objected to his sketch on the blackboard, saying that they would not draw the number line in the way he had drawn it. By this time he was alert to questioning their perceptions, and wanted to ask them to explain. However, he did not have a tape recorder available. So he diverted the focus of the lesson elsewhere to preserve the possibility of asking them to talk about their images at a later date. He transcribed the subsequent interview and presented his findings and perceptions at a seminar at the university to gain feedback on his analysis of the event.

This example illustrates that, at one level, Sam's enquiry could be seen as rather *un*systematic. He did not have a well defined research question with a clear methodological approach to answering it. However, he was aware of an aspect of his practice which he wanted to explore, namely the relationship between his expectations of students and their response to the tasks he set them. Initially, the only way he could think of to explore this was to have someone observe *for* him. However, these observations threw up further questions. It then seemed appropriate to interview students to try to find out more about their perceptions of his teaching and their feeling about the mathematical tasks in which they engaged. This deeper enquiry led to the exposure of students' mathematical insights about which he might otherwise not have become aware.

The process which I have described might be seen as one of evolving methodology as Sam explored, reviewed, discussed and explored further. Each stage of the exploration took him deeper into the perceptions and understandings of the students, providing insights for him of their views of and reactions to his teaching. As a result he began to see necessity for changing his teaching. This evolution of the research process was a dynamic which sustained motivation for further research. The research almost gained a momentum of its own.

At project meetings Sam described his process of enquiry: what he had done, the thoughts and questions which had arisen from doing it, and issues related both to the substance of the research and his methodology. From responses and questions of other teachers, and accounts of their research, he was able to gain a wider perspective of the issues involved and refine his own thinking. During the first year of the project, most teachers were happy to talk about their process of enquiry and what they learned from it. However, it was at the end of the year, when the pilot study was coming to an end and we were discussing the future of the project, that teachers admitted explicitly to feeling of insecurity with the research process. They said, "We cannot stop now, our research is only really just beginning". Implied was that they had been struggling with what research might actually mean for them, and had now developed an awareness that it was something within their control with which they were able to engage overtly and confidently. They might now go on with it more knowledgeable and less intuitively than in the past. Interestingly, several teachers are now beginning to write about their research with journal articles in mind.

The substance of a teacher's research

For Sam, the substance of his enquiry seemed fundamentally to be how he would achieve in practice his theoretical aims for the way he wanted his students to work. An example from his classroom and subsequent discussion with the University researcher illustrates the quality of thinking, the questions with which he was struggling, and some insights into the nature of his process of enquiry.

He had set his class a task involving the numbers 6, 3, 2, in that order and the operations $+ - \times \div$. Students had to write down all the possible ways of placing two operations between the three numbers, and bracketing either the first pair or the second pair of numbers. He gave three examples to illustrate what he meant by this: i.e. $(6+3)+2$; $(6+3)\div2$; and $6+(3\div2)$. He felt that these gave an indication of the different types of combinations they might develop: they could have two operations the same, or different. They could bracket the first pair of numbers or the second pair. He recognised that students would be successful at the task if they were systematic about it, but wanted them to realise this for themselves. When students asked how many cases there would be, he reflected the questions back to them: how many did they expect? It emerged that there were 32 possible ways. He then asked two further questions: Can you justify why there are only 32 combinations? and When are brackets unnecessary?

It appeared that the final question was "the point of the lesson" – what he hoped would be achieved through the students' activity. He wanted students to go on to recognise the forms of algebraic expression which could be written without brackets, and where brackets were essential. He could of course have explained this to the class in far less time than it took for them to tackle the task and subsequent questions. So, why did he want them to go through this process?

He said, "I want them to do something and then reflect on what they've done", and "I want activities that will challenge them to think". Although recognising that they needed to be systematic to be successful, he did not want to tell them to be systematic. It was something they needed to become aware of themselves. He did not write up the final pair of questions on the board until some of the class had got to a point in their exploration, that they needed the questions in order to go further. Some could possibly have achieved the aim of the activity without being given these questions. The questions could have arisen from their exploration and thinking. Thus making explicit the questions, reduced their opportunity to come to the questions themselves. However, other students did need to be given the questions, and might not have come to the questions otherwise. Some students complained that the teacher had set a second task in the questions on the board, and that only the better students in the class would be able to tackle this new task.

Sam grappled with what teaching actions suited both (a) his aims and expectations for his students' thinking and achievement, and (b) his students' perceptions of their needs in working on the tasks. His reflections on (b) caused him to rethink the implementation of his aims. Earlier the same morning, I had, at his request, interviewed a number of his students about their perceptions of mathematics lessons and Sam's expectations of them. One girl maintained that Sam gave most of his attention to the better students in the class, mainly boys; that activities were designed for the achievement of these students. Sam recognised that the boys mentioned could probably have achieved the aims of the lessons without being offered the final questions. It was for students such as this girl that the questions were made explicit. However, she saw the questions being an extra task designed by the teacher for the better students in the group, and as making her own achievement less secure.

The student interviews allowed Sam to gain insight into his students perceptions and to set these against his own aims and decisions in designing the tasks and asking the questions. There was evidence of considerable tension for him as his actions seemed to be misinterpreted. Despite seemingly sound classroom decision-making, the results ran counter to his best intentions. He recognised the personally threatening nature of these findings, and the need to know more about students perceptions in order to reconcile outcomes and aims.

Thus the cycle of his research involved reflection on his design of tasks and classroom outcomes, alongside gaining knowledge of students' perceptions and responses to the tasks set. This enquiry had a self consistency, not always evident in the somewhat haphazard way in which information was sought.

Mathematics was central to Sam's research. His own view of mathematics, encompassing processes and ways of thinking inseparable from whatever mathematical content was the focus of study, guided his design of teaching. Students' perceptions of mathematics, as well as Sam's approaches to teaching, influenced strongly the resulting mathematical achievement. The length of this paper restricts presentation of further evidence of these claims, although much such evidence exists from interviews and recording of student talk during lessons.

The pilot study indicated a rich potential of this kind of enquiry for the development of teaching by the teachers concerned. However, many counter-indications were also recorded. For example, teachers showed considerable dependence on the university researchers to sustain their research. There were many external pressures which militated against steady and continuous enquiry. Nevertheless, other support mechanisms started to emerge, for example in teacher collaboration within and between schools. Project meetings were a source of renewal and motivation. The next phase of the project will look more closely at such support mechanisms.

The involvement of the university researchers was itself also a source of enquiry relative to the findings of the project. Acting variously as researchers and as advisors to the teachers it was not always possible to separate these roles. While trying not to influence overtly the substantive research of the teachers, it was inevitable that influences arose from the researchers' presence, from the questions they asked, and from their known opinions and views. They were not strangers to the teachers. The project report will include an analysis of their roles.

All members of the project were strongly aware of the imporance of *mathematics* to the enquiry, and also that this was not always obvious in the discussions and analyses taking place. Further analysis will be focused explicitly on linking the findings of the research to the fact that it is *mathematics* teaching and learning which is under scrutiny.

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USING CHILDREN'S PROBABILISTIC THINKING TO INFORM INSTRUCTION

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This study developed and evaluated an instructional program in probability for grade 3 children. The instructional program was informed by a cognitive framework that described children's probabilistic thinking. Two classes participated in the instructional program, one in the fall and the other in the spring semester. Following instruction, both groups displayed significant growth in probabilistic thinking that was not simply due to maturation. There was evidence from four case studies that students' readiness to focus on the set of possible outcomes in a probability situation, their ability to connect sample space and probability, and their predisposition to use number in describing probabilities were key factors in facilitating learning.

Recent recommendations recognize the importance of having all children develop a greater awareness of probability (e.g., Australian Education Curriculum Corporation, 1991; Department of Education and Science and the Welsh Office, 1991; National Council of Teachers of Mathematics, 1989). This emphasis on probability in the school curriculum has established the need for further, on-going research into the teaching and learning of probability (Shaughnessy, 1992). In relation to teaching and learning, Fennema, Franke, Carpenter, and Carey (1993) call for instruction that is informed by research-based knowledge of children's mathematical thinking. Although there has been considerable research into children's thinking in probability (see Shaughnessy, 1992), none of the research has generated or evaluated instructional programs that are guided by research-based knowledge of children's probabilistic thinking.

This study addresses the development and evaluation of such an instructional program. In particular, it seeks to: (a) use a framework that describes and predicts children's thinking in probability to construct a third-grade instructional program; and b) evaluate the effect of two different sequences of the instructional program on children's thinking in probability.

Theoretical Considerations

The instructional program developed in this study is based on a cognitive framework that describes children's probabilistic thinking (Jones, Langrall, Thornton, & Mogill, in press). This framework provided the research base for informing the instructional program and constructing assessment protocols.

CONSTRUCT	Level 1 Subjective	Level 2 Transitional	Level 3 Informal Quantitative	Level 4 Numerical
SAMPLE SPACE	• lists an incomplete set of outcomes for a one stage experiment	• lists a complete set of outcomes for a one-stage experiment, and • sometimes lists a complete set of outcomes for a two stage experiment using limited and unsystematic strategies	• consistently lists the outcomes of a two stage experiment using a partially generative strategy	• **adopts and applies a generative strategy which enables a complete listing of the outcomes for a two- and three-stage case**
PROBABILITY OF AN EVENT	• predicts most least likely event based on subjective judgments • recognizes certain and impossible events	• predicts most least likely event based on quantitative judgments but may revert to subjective judgments	• predicts most least likely events based on quantitative judgments including situations involving non-contiguous outcomes) • uses numbers informally to compare probabilities • distinguishes "certain," "impossible," and "possible" events, and justifies choice quantitatively	• predicts most least likely events for single stage experiments • assigns a numerical probability to an event (it may be a real probability or a form of odds)
PROBABILITY COMPARISONS	• compares the probability of an event in two different sample spaces, usually based on various subjective or numeric judgments • cannot distinguish "fair" probability situations from "unfair" ones	• makes probability comparisons based on quantitative judgments (may not quantify correctly and may have limitations where non-contiguous events are involved) • begins to distinguish "fair" probability questions from "unfair" ones	• makes probability comparisons based on consistent quantitative judgments • justifies with valid quantitative reasoning, but may have limitations where non contiguous events are involved • distinguishes "fair" and "unfair" probability generations based on valid numerical reasoning	• assigns a numerical probability measure and compares • incorporates non contiguous and contiguous outcomes in determining probabilities • assigns equal numerical probabilities to equally likely events
CONDITIONAL PROBABILITY	• following one trial of a one stage experiment, does not give a complete list of outcomes even though a complete list was given prior to the first trial • recognizes when certain and impossible events arise in non-replacement situations	• recognizes that the probability of some events changes in a non-replacement situation, however recognition is incomplete and is usually restricted only to events that have previously occurred	• can determine changing probability measures in a non replacement situation • recognizes that the probabilities of all events change in a non-replacement situation	• assigns numerical probabilities in replacement and non-replacement situations • distinguishes dependent and independent events

Figure 1. Framework for Assessing Probabilistic Thinking

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Framework for Children's Probabilistic Thinking

In order to capture the manifold nature of probabilistic thinking, our framework (Figure 1) incorporates four key constructs: sample space, probability of an event, probability comparisons and conditional probability. The framework builds on previous research in sample space (e.g., Borovcnik & Bentz, 1991; English, 1993), in probability of an event (e.g., Acredelo, O'Connor, Banks, & Horobin, 1989; Fischbein, Nello, & Marino, 1991), in probability comparisons (e.g., Falk, 1983), and in conditional probability (e.g., Borovcnik & Bentz, 1991). However, our framework is distinctive in that it provides a coherent and comprehensive picture of children's probabilistic thinking that enables benchmarks for instruction and assessment to be established.

For each of the key constructs (Figure 1), four levels of thinking were established and validated. Level 1 is associated with subjective thinking, Level 2 is transitional between subjective and naive quantitative thinking, Level 3 involves the use of informal quantitative thinking, and Level 4 incorporates numerical reasoning. These levels of thinking evolved from observations of children's probabilistic thinking over a two-year period and appear to be consistent with neo-Piagetian theories that postulate the existence of levels of thinking that recycle during developmental stages (Biggs & Collis, 1991; Case, 1985).

Methodology

Subjects

Students from two intact Grade 3 classes participated in an instructional program in probability—one class in the fall semester (Early Instruction Group, n = 18), the other class during the spring semester (Delayed Instruction Group, n= 19). In addition to the analysis involving all students, two children from each classroom were randomly selected as case studies for more detailed analysis.

Procedure

Each semester's instructional program consisted of sixteen, 40-minute sessions--two per week over eight weeks. Sessions opened with a whole-class exploration posed by one of the researchers. Twelve teacher education student mentors then worked with pairs of children to solve probability problems.

Instructional Program

The instructional intervention, developed by the researchers (Jones & Thornton, 1992), consisted of problem-driven tasks generated from the four key constructs of the Probabilistic Thinking Framework. The pedagogical orientation of the intervention was grounded in research-based knowledge of children's probabilistic thinking, and was based on learning within a socio-constructivist environment (e.g., Wood, Cobb, Yackel & Dillon, 1993). Mentors participated in weekly seminars to explore ways to 1) use the framework to assess and build on children's understanding; 2) pose problems rather than model solutions; 3) guide students to construct their own solutions; 4) maximize opportunities for students to collaborate; and 5) challenge students to negotiate problem solutions.

Data Collection, Instrumentation and Analysis

Interview and observational data were gathered from three sources: 1) researcher-designed assessments conducted at the beginning, middle, and end of the school year; 2) mentor evaluations from each instructional session; and 3) researcher field notes on the case study students. The assessment protocol based on the Probabilistic Thinking Framework comprised 20 tasks: five on sample space, four on probability of an event, seven on probability comparisons, and four on conditional probability (see Figure 2).

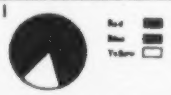
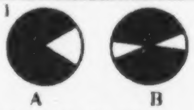
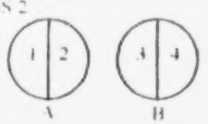
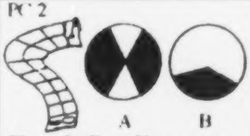
Sample Space	**Probability of an Event**	**Probability Comparisons**	**Conditional Probability**
SS 1 I will shake this box *Allow the student to observe that this box contains 4 green, 3 red, 2 yellow bears* If you close your eyes and draw a bear from the box, what colors could your bear be? Why?	PE 1  This spinner is used to play the penny game You and a friend pick a color and then take turns spinning If the pointer lands on the color you picked, you get a penny If not, you lose a penny Which color would you choose? Why?	PC 1  We will play the penny game again (See PE 1) Which spinner would be the best for you? Why? *If child prefers one spinner over the other, ask,* Does it matter which spinner you use? Use numbers to tell me about the chances of getting a red	CP 1 (continuing from PE 2) *Two red bears are drawn and not replaced* I will shake the box again If you draw another bear without looking, what color would you have the **LEAST** chance of getting? Why?
SS 2  Spin both spinners What numbers did you get? What numbers could you get if you spin both spinners again and again? Is that all? How do you know?	PE 2 I will shake this box *Allow the student to observe that the box contains 4 green, 3 red, 2 yellow bears* If you draw a bear without looking, which color do you have the **LEAST** chance of getting?	PC 2  *Place the Race Home Game mat before the student* Show the spinners Your color is red Which spinner would be best for you if you wanted to win? Why? Can you use numbers to explain your choice or compare the chances on the 2 spinners?	CP 2 (continuing from CP 1) Has the chance of getting any of the colors changed since we began drawing bears? which colors? Why? Can you use numbers to explain?

Figure 2: Interview assessment: Selected items

Two different procedures were utilized to code the interview assessments. The first procedure, used with the four case study students, involved double coding (Miles & Huberman, 1984) to establish probabilistic thinking levels on each of the four constructs over the three assessment points. For details of this procedure see Jones, et al. (in press). The second procedure, which also involved double coding, was used to generate performance scores (maximum = 20) for all students in the Early and Delayed Instruction groups over the three assessment points. Working independently, two of the authors scored children's responses with 92% agreement. Variations were clarified until consensus was reached. Qualitative data on children's probabilistic thinking was collected by the mentors and the research team. Each week mentors recorded insightful, interesting, or unusual responses--to develop a profile of each child's probabilistic thinking.

An analytic inductive theory approach (Erickson, 1986) guided the analysis of qualitative data on the four case study students. Transcripts of the mentor evaluations and researcher field notes were coded and synthesized to discern learning patterns exhibited by these students during the intervention. A repeated measures analysis of variance was also performed on the data collected at three assessment points for both the Early Instruction and Delayed Instruction groups.

Results

The Effect of the Intervention Program: Analysis of Case Studies

Two of the case studies, Jana and Kerry, were typical of students whose thinking in sample space developed very slowly and substantially restricted their overall growth in probabilistic thinking. In spite of the intervention, both students seemed unwilling to recognize or identify all possible outcomes in a one-stage experiment and were unable to build a systematic strategy for listing two stage outcomes. Even when Kerry's sample space thinking matured toward the end of the intervention she was not inclined or not able to make connections between the composition of the sample space and the probability or conditional probability of events in that sample space.

By way of contrast, Corey and Deidra whose thinking profiles were similar to that of Jana and Kerry prior to the intervention, showed strong and consistent growth across almost all constructs. While Corey and Deidra's rapid growth in sample space thinking was an important factor, a more crucial factor was their predisposition to describe and justify probabilities using quantitative and or numerical reasoning. Deidra, in particular, was able to use fractions in a meaningful way to explicate her thinking and as a result was exhibiting thinking beyond level 3 on three of the four constructs by the end of the intervention.

The Effect of the Intervention Program: Analysis of the Instructional Groups

The children's probability performance on the interview assessment items was analyzed for each of the three assessment points. The relevant means and standard

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deviations for both the Early Instruction Group and the Delayed Instruction Group are presented below.

Table 1

Means and (Standard Deviations) for Pre, Middle, and End Assessments

Group	Beginning	Middle	End
Early Instruction (n=18)	12.44 (2.56)	15.39 (2.70)	15.00 (2.47)
Delayed Instruction (n=19)	13.11 (2.77)	13.95 (2.32)	16.63 (1.54)

A repeated measures analysis of variance performed on the data for the three assessment points indicated significant differences for the assessment points ($F=12.88$, $p < .001$) and a groups by assessment points interaction ($F=6.38$, $p<.01$). Further analysis using the Tukey-HSD test (Kirk, 1982), showed that the interaction was produced by non parallel differences at assessment points within each group and by non parallel differences between groups (see Table 1).

In the case of non parallel differences *within* each group, the Early Instruction Group showed significant differences between beginning and mid assessments ($p < .01$) and between beginning and end assessments ($p < .01$)—but not between mid and end assessments (see Table 1). On the other hand, the Delayed Instruction Group showed significant differences between the end assessment and each of the earlier assessments (in each case, $p < .01$). Non parallel differences *between* the Early Instruction Group and the Delayed Instruction Group were evident in significant effects at both the mid and end assessments (in each case, $p < .05$). Moreover, at both the mid and end assessment points, the differences between the groups were reversed from the previous assessment (see Table 1).

Discussion

This study addressed a need for the development and evaluation of instructional programs in probability that are informed by research-based knowledge of children's thinking. In particular, the instructional program utilized in this study was based on a cognitive framework that described and predicted children's probabilistic thinking (Jones, et al., in press).

With respect to the effectiveness of the instructional program, a repeated measures analysis of variance demonstrated that both the Early and Delayed Instruction groups showed significant growth in performance following instruction. Moreover, because the Delayed Instruction group essentially acted as a control group between the first and middle assessment points, the significant difference in favor of the Early Instruction group at the middle assessment point provides further evidence that the instructional intervention generated learning that was not simply due to maturation. Notwithstanding the overall effectiveness of the instructional intervention, the size and consistency of the standard deviations for both groups at almost all stages of the study (See Table 1) predicate substantial variation in the probabilistic thinking of children involved in this study. Some insights into these variations were revealed from the case study analyses involving Jana, Kerry, Corey and Deidra. The differential effect of the instructional program appeared to be linked to three discernible learning patterns: children's initial level of thinking in sample space; their willingness to connect sample space and probability; and their predisposition and ability to use number and fractions in describing probabilities and conditional probabilities.

Interestingly, in the four-month period following the intervention, Jana showed substantial growth in probabilistic thinking. It is not clear whether her growth in probabilistic thinking was a delayed effect of the instructional program, or whether it was indicative of more general developmental growth. We believe that Jana's sudden spurt and the Delayed Instruction group's superior performance at the end of the study provide evidence of the importance of maturational development in the timing of probability instruction. At any rate, both the quantitative and qualitative analyses associated with this study suggest the need for further research to investigate the long-term effects of delaying instruction in probability. Given the impressive growth in probabilistic thinking exhibited by students who had a predisposition to use numbers, especially fractions, there may well be considerable merit in delaying the introduction of probability until students have greater "number power."

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COMING TO KNOW ABOUT 'DEPENDENCY' WITHIN A DYNAMIC GEOMETRY ENVIRONMENT

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The ability to define relationships between objects is one of the most powerful features of a dynamic geometry package such as Cabri-Géomètre. *In this paper I document how one pair of 12 year old students begin to come to know about this form of functional dependency within this particular computer environment. I suggest that this process of coming to know about dependency may be understood as an interweaving between the 'voices' of the students and the teacher within the socially organised activity taking place in the classroom.*

La capacidad para definir relaciones entre objetos es una de las caracteristicas mas potentes de un paquete integrado de geometria dinámica como es el Cabri Geometria. *En este artículo, presento como un par de alumnos de 12 años empiezan a comprender el concepto de dependencia funcional dentro de este contexto informático particular. Sugiero que este proceso de iniciación a la comprensión de la dependencia puede interpretarse como un entretejerse las 'voces' de los alumnos y el profesor dentro de una actividad socialmente organizada que tiene lugar en el aula.*

Introduction

One of the most powerful features of a dynamic geometry package such as *Cabri-Géomètre* is the ability to define relationships between objects and to explore graphically the implications (Laborde 1993 p 53). The drag facility allows a figure to be continuously transformed while the relationships between the objects remain invariant. The idea of dependency (and independency) can be explored by, for instance, observing the nature of the relationships between the objects used to construct the figure in question when a chosen object such as a point is dragged. Another way of observing dependency is when an object is deleted. Then all dependent objects are also deleted.

As Holzl *et al* (1994) discovered when they observed pupils attempting to construct a rectangle, the students had to come to terms with "the very essence of Cabri; that *a figure consists of relationships* and that there is *a hierarchy of dependencies*" (emphasis in original). An example of this hierarchy of dependencies is the difference (in *Cabri I* for the PC) between *basic point*, *point on object* and *point of intersection*. While all three types of point look identical on the screen, basic points and points on objects are moveable (with obvious restrictions on the latter). Yet a point of intersection cannot be dragged. This is because a point of intersection depends on the position of the basic objects which intersect. In their study, Holzl *et al* found that students need to develop an awareness of such functional dependency if they are to be successful with non-trivial geometrical construction tasks using *Cabri*. The experience of Holzl *et al* is that "Not surprisingly, the idea of functional dependency has proved difficult [for students] to grasp".

In this paper I describe how one pair of 12 year old students begin to come to know about dependency within the dynamic geometry environment *Cabri*. The data comes from a project designed to trace the transition of student conceptions of some chosen geometrical objects from informal notions towards formal mathematical definitions. I begin with an outline of the theoretical framework with which I will interpret the data.

Theoretical framework

It is evident that "coming to know" is a complex process and that an understanding of such a process cannot be explored in a framework that detaches that learning from its sociocultural setting. Mercer (1995 and with Edwards 1987), for instance, has expounded on the guided construction of knowledge within the classroom by stressing the importance of talk between teachers and learners. Wertsch (1991), too, has built on the work of Vygotsky and others with the claim that "human action typically employs 'mediated means' such as tools and language, and that these mediated means shape the action in essential ways" (p 12). Yet as Confrey (1995a) points out, there are a number of limitations to employing an overly narrow Vygotskian perspective (or its interpretation) in mathematics education. These include:

1. Vygotskian theory (or its interpretation) may encourage the neglect or devaluation of concrete activity
2. Advocates (or interpreters) of Vygotskian theory may focus on, and privilege, language to the detriment of other forms of intellectual interaction and inquiry.

Indeed as Cobb (1993 and 1995) has shown, classroom learning of mathematics is not always consistent with the sociocultural view that social and cultural processes drive individual thought. Nevertheless, both Confrey and Cobb point to ways of moving beyond the tensions that are apparent between a Piagetian (individualistic) and a Vygotskian (sociocultural) viewpoint. They point to an interweaving of a student's own cognising activity within the socially organised activity in which the student is a participant. As Cobb (1995) says "it is impossible to understand how students could construct an intellectual inheritance that took millennia to create unless we understand how their negotiation and use of symbolic means supports their mathematical development".

Confrey (1995b) employs a distinction between 'voice' and 'perspective' to signal the two kinds of learning that result from a reciprocal interaction between a student and a teacher (reciprocal in that both parties learn). 'Voice' refers to the student's conceptions while 'perspective' can be used to describes the teacher's viewpoint. This has echoes with Wertsch's (1991) concept of the 'voice' of the mind and how learning through talking and thinking involves 'ventriloquating' through the voices of others. These ideas are based on the work of Bakhtin who stressed that voices always exist in a social milieu so that there is no such thing as a voice that exists in total isolation from other voices. 'Ventriloquating' is the process whereby one voice speaks *through* another voice. As a student begins employing a term such as 'dependency' it is initially only half theirs. "It becomes one's own only when the speaker populates it with his own intention, his own accent, when he appropriates the word, adapting it for his own semantic and expressive intention" (Bakhtin 1981 pp 293-294).

Confrey (1995b) proposes that "classrooms can be described as places in which children engage in grounded activities and systematic enquiry". Grounded activities are, according to Confrey, "actions involving practical activity which are mediated by one's interactions with others". In contrast, systematic enquiry involves "communication through the use of signs" which "can be viewed as social activity mediated by one's experience in grounded activity". Confrey suggests that "looking at the interactions between these two forms of mediated activity may yield some useful insights into how we might successfully educate people in mathematics".

In the case study that follows, I document how one pair of 12 year old pupils interact with the teacher/researcher regarding the notion of dependency during four 50-minute mathematics lessons that took place at intervals over a period of six months. I suggest an interpretation of the data from this case study making use of the notions of the interweaving between voice and perspective and of the process of 'ventriloquating'.

The Case Study

The pair of students reported on here are 12 year olds with no previous experience of using a dynamic geometry package although they have used various drawing packages and other IT resources. The class is an above-average mathematics class in a city comprehensive school whose results in mathematics at age 16 are at the national average. The mathematics teachers use a resource-based approach to teaching mathematics and the students usually work in pairs or small groups. The class has three 50-minute mathematics lessons per week. For this part of the study, computer use for *Cabri* was restricted to one computer in the classroom (the students have access to computer laboratories for other computer applications). This meant that, as student pairs took it in turn to use the computer, it was often several weeks between sessions for particular pairs. The version of *Cabri* used was *Cabri I* for the PC.

In an introductory session the pair of students were introduced to some of the menu items in *Cabri* and then allowed to choose their own goal. The notion of "messing up" (a term suggested by Healy *et al* 1994 to refer to whether a figure could be dragged to see if it became unrecognisable) was introduced, with the students being encouraged to formulate *mathematically* challenging goals. Following this introductory session, the students worked through a series of tasks on quadrilaterals (Jones 1995). Each of the classroom tasks required the students to analyse a figure presented on paper and to construct the figure using *Cabri* such that the figure is invariant when any basic point used in the construction is dragged. This means that the students have to focus on the *relationship* between the basic objects (points, lines and circles) necessary to construct the figure.

The Exploratory Session

There are three explicit references to the notion of dependency during the initial exploratory session. The first comes from the students when they have created a circle by centre and radius point. They find that dragging the centre point changes the size of the circle. I ask them what will happen if they drag the radius point.

C: It [the circle]will get smaller or bigger depending which way you moved it [the radius point].

This indicates the students have some idea of functional dependency. Later in the session, during the drawing of a 2D representation of a cube (which they refer to as a box), the students want to delete a point. When attempting to do so they get the following message from *Cabri*: "Delete this object and its dependents?"

C: Dependents? Is that the whole box [ie cube]?
Me: Why don't you see, because you can undo it.

They delete the point and two line segments are also deleted. This gives me the opportunity to explicitly refer to dependency.

Me: So that bit of line depended on that point, and that bit of line did, so they both went.

Near the end of the lesson, the students construct a triangle and its three angle bisectors. They construct points of intersection but find that these points cannot be dragged.

Me: These points [pointing at the points of intersection] depend on these points [the points used to create the triangle].

After a little thought and dragging, one of the students says:

C: You can't drag that point [a point of intersection] because it is dependent on them [indicating the points used to create the triangle].

Session 2

In session 2 the students complete two tasks involving lines and circles. At one point, one of the students asks:

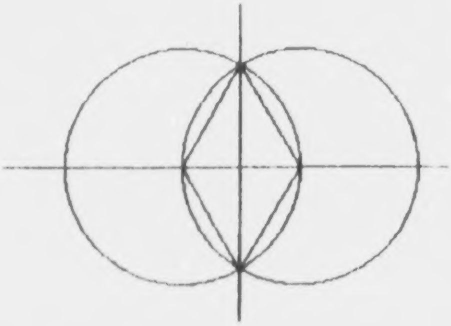
C: What's the intersection doing? Does it keep the dot [the point] there?
Me: What you are finding is the point *here*, where the circle crosses the line.
C: Right, so if it was like *that* [indicating a different arrangement of lines], then it [the point of intersection] would be *there*.
Me: It is always where the lines cross.

(note that, in this exchange, I did not mention dependency. I will comment on this later in the paper). The students complete the task and I ask them why the figure cannot be "messed up". One of the students replies:

H: They stay together because of the intersections.

Session 3

During this session the students are asked first to construct another pattern of circles and then to construct the figure given below.



Referring to points of intersection, one of the students comments:

H: A bit like glue really. It's just glued them together.

A little later, the other student asks why you can't drag points of intersection.

Me: Because the intersection points just show you where two things cross.
C: So how come it keeps it together if it's just a dot to show you where they cross?

Me: You can move *that* point because it's the centre of the first circle that you drew.
So if you move that, then because you are changing the size of the first circle, the point where it crosses the other circle changes so that changes the other circle.
H: So that changes everything.
Me: Because the other circle depends on that.
C: So because it depends on it, it moves.

Session 4
The students successfully construct the required figure during this session and explore, with me, which objects can be dragged and what is the effect of dragging them.

C: So it's all about depending on stuff, isn't it?
Me: It's like a function. When one thing is a function of another it depends on the other.
C: So there's a rule in Cabri ....
... if things don't depend on each other you have to make them depend on each other to know what moves, because ..
H: to make...
C: so everything ...
H: to make the pattern
C: depends on one thing
H: to make the pattern and then it's non-messupable.
C: and then it can move. But because everything is dependent on one thing then it will always be the same, related to each other.

Discussion
The above extracts of student/teacher dialogue illustrate how one pair of students began to come to understand the notion of dependency within the context of the dynamic geometry package *Cabri*. They begin with an existing notion of dependency, knowing, for instance that the size of a circle depends on its radius. They also readily understand that when an object is deleted its *immediate* dependents are also deleted. As the students encounter points of intersection and need to construct objects dependent on these points, hence creating chains (or hierarchies) of dependency, then a way of explaining what is going on can be based on the theoretical framework introduced earlier.

This explanation involves viewing the interaction as an interweaving between individual sense-making and the social situation of a pair of students jointly working on a task and being able to refer to me whenever they thought it necessary. This interweaving is so strong that it is probably unwise to attempt to separate out, to too great an extent, any of the individual constituents (Wertsch refers to this as speaking of "individual(s)-acting-with mediated means" 1991 p 12). In this way, the case study can be viewed as an example of the interaction between the two forms of mediated activity (grounded activity and systematic inquiry) proposed by Confrey (1995b). There is a sense in which the students are both borrowing terms from Cabri (for example, dependents) and modes of expression from me *and speaking*, at least in the early stages, *as if they were me*. For example, the statement by one of the students towards the end of the initial exploratory session can be interpreted as an instance of 'ventriloquating'.

C: You can't drag that point [a point of intersection] because it is dependent on them [indicating the points used to create the triangle].

In the second session, when the students ask for clarification of the nature of a point of intersection, I do not refer to notions of dependency. I merely state that a point of intersection "is where the lines cross". As it transpires, the students have developed their own interpretation of the nature of points of intersection.

H: A bit like glue really. It's just glued them together.

Ainley and Pratt (1995) have noted the same sort of interpretation of points of intersection. During session 3, I become aware of the students' interpretation and this time I do refer explicitly to dependency. During session 4, it is the students who raise the issue of dependency. By this session they seem to be recognising its central importance and are beginning to offer their own explanation of dependency. In Bakhtian terms, it could be said that the students are beginning to populate the notion with their own intentions. In terms of Confrey's (1995b) notions, the students' solving of some geometrical problems can be viewed as "grounded activity" while their coming to know about dependency is "systematic inquiry".

In this paper I only document the explicit, and necessarily mostly verbal, uses of the notion of 'dependency'. Nevertheless, these explicit references combine verbal statements with practical activity in a way that cannot be separated. The verbal statements all refer to *action*. Wertsch (1995 p 71) maintains that "some notion of action holds the key to avoiding potential dead ends in sociocultural research", although he admits "I am less certain that the notion of mediated action I have outlined here [and in Wertsch 1991] will ultimately fill the bill".

There is, in addition, within the sessions briefly described in this paper, a continuous movement between teacher/researcher goals and student-orientated goals. For example, in the initial session the students are able to choose their own goals but for me these had to be *mathematically* challenging goals. In later sessions, although tasks were set, sub-goals were student-chosen and interventions were kept to a minimum.

Finally, throughout the sessions with this case study pair, there are also numerous examples of the implicit use of the idea of dependency. These are, for the most part, captured on videotape. It may be that when an analysis of these is added to the account, a fuller picture of the interweaving between the 'voices' of the students and the teacher within the socially organised activity taking place in the classroom will result.

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SOMALI CHILDREN LEARNING MATHEMATICS IN BRITAIN: A CONFLICT OF CULTURES.

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Summary

Five children in the reception class of a London school were observed at school and at home. Interviews were carried out with their teachers in school and their parents at home. Data was collected on the assumptions made by parents and teachers about the children's learning in the different domains, the materials used in the acquisition of numeracy and the methods employed in the teaching and learning processes. The study suggests that the approaches taken by parents and teachers are very different, with the most formal learning situations taking place in the home. The recognised difficulty of effecting learning transfer suggests that teachers may need to take more account of the children's home learning experience.

Cinco niños en su primer año de estudios en un colegio de londres fueron observados en el colegio y en casa. Se llevaron a cabo entrevistas con sus profesores y padres. Se recabo información basada en los detalles aportados por los padres y profesores de los niños. Tambien se consideraron los materiales y metodas utilizados en el proces. De aprendizaje mientras los alumnos adqirian las diferentes habilidades necesarias para trabajar con números. El estudio sugiere que los modos en que padres y profesores acomenten la tarea de enseñar a los niños son muy diferences y que la enseñanza más formal tiene lugar en casa. La dificultad reconocida de conseguir una 'transferencia de aprendizajes' sugiere que quizás se debe considera aún más qué tipo de enseñanza están los niños recibiendo en casa.

Introduction

Learning transfer

The theoretical framework for the research arises from the view that children learn mathematics through constructing their own knowledge in a process of interaction with their environment. Ernest clarifies different views of social constructivism (Ernest, 1994) and the research described here fits into the Vygotskian framework which he identifies. There is some evidence that knowledge is not only socially constructed, but is also 'situated' (Brown, Collins & Duguid, 1989). The notion of situated cognition suggests that the actual cognitive processes which bring about learning occur within a specific context. There is a gathering body of evidence that there is little transfer of learning which takes place. Lave (1988) for instance, describes in her Adult Maths Project occasions when the adults

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use calculation methods which are entirely contextualised. The procedure and the cognitive process arises strictly in relation to the context in which the learner finds himself or herself. Nunes, Schliemann & Carraher (1993) provide examples of children working in street markets in Brazil. These children were found to be considerably more successful at mental calculations carried out in the market context than when they attempted to perform the same calculations by means of a standard algorithm. The methods used by the children out of school are not just reliant on memorised facts or number bonds. They were asked to charge for purchases which required them to perform mental calculations and their commentary makes it clear that calculations were indeed necessary. One of the most worrying aspects of this research is that when children are placed in a school context they abandon successful strategies learnt and used in the 'real world' and (presumably) feel obliged to use school methods, regardless of the fact that these are frequently unsuccessful for them.

The involvement of parents in children's learning

In Britain there has been a tradition of parents' involvement in children learning to read. Schools frequently have a system by which children take reading books home and parents with young children at school are expected and encouraged to hear their children read. The practice became more widespread and more formalised following the publication of a number of research studies indicating its effectiveness. (e.g. Tizard, Schofield and Hewison, 1982, Hannon and Jackson, 1987, Griffiths and Hamilton, 1984)

In mathematics teaching there was no equivalent tradition of involvement of parents, particularly with the learning of young children starting school. There is, however, one curriculum project in Britain which has set out to encourage parental involvement in children's mathematical learning (Merttens & Vass, 1990). Morgan and Merttens (1994) claim that this scheme is, "Not socially divisive in the fashion of traditional homework." and that, "all they (parents) are required to do is to support their child's learning to talk through a task or to act as a resource." However, what this project seems to offer is the chance for parents to share the school culture and the school view of what maths learning involves, but very little chance for the home culture or understanding to make an impact on school practices.

Hughes et al. (1994) set out to find out what parents actually want from their children's education. One of the main findings was that parents knew little about what their children were learning in school. They wanted to know more but despite the fact that they had plenty of contact with the school, this contact did not provide them with the information they needed and they relied on their children as the main source of information. This result was mirrored in the present study with the Somali families. The parents who worked at home with their children questioned them about the mathematics they did at school and tried to keep one step ahead of the number work covered in school. They did not feel that they were kept well informed by the school.

Aims of the study

The study aimed to identify numeracy practices taking place in the home and school environment for this group of children and to promote partnership between parents and teachers in which different approaches to numeracy are recognised. Many parents have ways of working with their children and supporting their learning which are not recognised or valued by teachers. Rogoff and Lave (1984) found that knowledge gained from the home is often discounted in school.

Methodology

Although there has been a considerable influx of Somali refugees to London over the last few years, it proved surprisingly difficult to locate a school with more than one or two children in the reception or Year 1 classes. The school chosen for the project was the only one in which it was possible to locate a large enough cohort of Somali children in the target age range.

Ethnographic methodology was used in the study. One of the researchers is of Somali origins and was able to undertake the home visits through the medium of the Somali language. Participant observations were made in school and in the children's homes. Unstructured interviews were conducted with parents and siblings. Teachers and other professional adults were interviewed in school. Interviews were not recorded. A written record was produced soon after the interview had taken place. A total of 12 school visits and 13 home visits took place.

Findings

The teachers assumed from their own observations of the children that the parents helped their children at home by giving them lists of words to learn and helping them to learn their numbers. They thought that home teaching took place as formal lessons. They were also under the impression that a number of families lived together in one house or flat. In fact each family is housed separately, in low rise council flats all in the same locality. There is a strong informal community network, so that parents or older siblings collect a number of children to take them home. Some, but not all of the parents teach their children at home. The work they described placed a great emphasis on memorising.

The role of older siblings appeared to be very significant in the young children's learning experience. It was common for the older children to set tasks for their younger siblings, asking them to recite their tables, and keeping a check on what they had previously learnt. Although the mathematical content is traditional, the atmosphere is of informal play and the method is very similar to that used when the children learn the Qur'an. At the age of four or five all children start to learn the Qur'an by heart. They learn this section by section, without sight of the text. Once they have memorised a section, they will recite it to the other children at home or to the teacher. This forms a two-way learning process; listening to other children reciting will, no doubt, reinforce their own learning.

The children's learning experience at school stood in some contrast to their experience at home. At home the researchers expected to find examples of children's games which encompassed some aspects of spatial activity or numeracy, e.g. games such as hopscotch or their equivalent. Zaslavsky (1973) describes the extensive involvement of different cultural groups in games with mathematical significance. In the event, there was no evidence of such games being part of the repertoire of this group of children. At school the atmosphere, as in most reception classes in Britain, was fairly informal. The day usually started with children gathered together in a carpeted area exchanging news via the teacher and being guided to their tasks for the morning. For much of the day they would then work in small groups or as individuals on a variety of tasks. Amongst the mathematics activities we observed were games such as 'snakes and ladders', construction activities with lego or 'octons', measuring, using cubes as non standard units and written calculations. It is perhaps only the last of these activities that would be recognised by the parents as mathematical. The classroom atmosphere during lesson time is so relaxed and informal that it is perhaps not surprising that some children consider that the purpose of school is "to play" (Gregory et al, 1993)

Discussion

Aubrey (1994) details the mathematical knowledge which children bring to school, showing that they are not only able to draw on specific number knowledge, but to,

> "switch from earlier and less formal strategies to formal procedures when prompted to do so."

Interestingly the young children in her sample showed a flexible combination of formal knowledge with invented strategies. They were introduced to activities in the form of story problems with make believe situations involving puppets, teddies, sweets, small-scale toys and pictures of everyday familiar objects. It is possible that the effect of this approach was to bridge the gap between the home and school experience.

There is a strong oral tradition in the Somali culture and much of the work at home with children reflected this. With the young children in the present study the emphasis was on memorisation. Where parents were working with children at home they expressed a wish to see examples of their written work in mathematics in order to support their work at home. In a British reception class it would be unusual for teachers to send home examples of children's written work. Children are sometimes encouraged to take art and craft items home, but it is much less likely that they would be encouraged to take home a written record of mathematics.

Teachers would normally expect such work to be available for parents to see on the occasion of parents evenings, but these evenings were difficult for the parents in our study as they were not confident in their spoken English and, at the time, no translation facilities were available for Somali for this school. Parents had considerable respect for teachers and did not perceive a need to visit the school and question the teachers about their children's education.

The children in this study had to make more than usually large adjustments on entering school. The home language was different. (Parents were keen to maintain the use of Somali as the home language.) They were accustomed to life in an extended family and we were given a graphic example of this by one child who when asked in class "How many people in your family?" replied, "Hooyaday (my mum) aabbe (dad) walaalkay (my brother) Ayeeyadey Xamar joogta (my grandmother in Mogadishu)..." In their homes there were none of the toys and games which would commonly be found in British homes. The opportunity to play with these items in school, did not provide for them, as it would for many children, a linking experience between home and school, but a new and different experience. The expectations of their behaviour at home and at school were very different, with an increase in autonomy in the classroom. This contrasts sharply with Tizard and Hughes' description of the children in their study who found that on entering school many of the decisions which children had previously taken themselves were now dictated by the school routine.

Bishop (1994) in an analysis of the research and development needed to resolve cultural conflicts, suggests that one way forward is to "culturise " the formal mathematics curriculum, i.e. restructuring the curriculum in relation to the local culture. In multicultural schools in the U.K. it would be unrealistic to attempt to design a curriculum which took account of all home numeracy practices, though the curriculum can be planned to include a range of materials from a variety of cultures. However, if children are gaining cognitive experiences which are very different from those experienced in school we need to find ways to build bridges between the two and gain "mileage" from the understanding which children bring to school.

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From Measurement to Conjecture and Proof in Geometry Problems

- Students' Use of Measurements in the Computer Environment -

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Abstract

When middle school students solve proof problems in the geometric computer environment, the use of measurements helps them to confirm the geometric properties they discover, and measurements provide strong evidence to convince them of these properties. At that time, students are involved in the problems very actively (Kakihana & Shimizu, 1993). The purpose of this research is to investigate how students' strategies shift from conjecture to proof when they utilize measurements in the geometric computer environment. Junior high school level geometry problems were given to five pairs of women's junior college students and their activities were videotaped and analyzed. Students' worksheets were also collected and analyzed. From the results, it was identified that (1) the use of measurements for conjecturing in pair conversations helped to explain logically the reasoning or to construct statements for formal proof and (2) the use of measurement helps to convince students of the logical reasoning in a statement of proof.

I Introduction

New environments where students can explore the properties of geometric figures and theories of geometry by using software such as *Cabri-Geometry* and *Geometer's Seketchpad* are emerging very rapidly in the field of mathematics education. The use of software helps students to conjecture the properties of geometric figures and theorems (Kakihana,1991, Chazan,1992, Schumann,1993). In the Japanese lower secondary mathematics curriculum, "proof" is regarded as very important, but many surveys on achievements have revealed students do not like mathematics because of "proof". Usually, "proof"("Syomei" in Japanese) means to write down a deductive reasoning process with mathematical symbols in a precise and standard manner. Sometimes students

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say "Now I can prove this problem because I remember how to write it". This tendency shows how students recognize "proof"("Syomei"). To overcome these problems, a new type of proof activity in the lower secondary geometry curriculum has been explored(Chazan,1991,De Villiers,1990,1991,Shimizu,S,1994). Contrary to the argument in *The Death of Proof* (Hogan,J.,1993), Hitotsumatsu(1994) commented that "the proof which is the emphasis of the formalism has died. However, the new period has come to think about the new type of proof that can be done by using computers". John Costello(1994) maintains that "the construction of proof is for convincing themselves and other people and the construction of mathematical knowledge is made by thought experiment."

In the computer environment "continuous variation of geometric figures" (Schumann, ibid.) and "measurements by using computers"(Kakihana & Shimizu,1994) had a significant effect on students' geometrical performance. The plausible new method of teaching and learning "proof" in geometry is to be developed. When students solve a proof problem, students rely on the result of measurements to convince themselves of the generality/validity of properties they discovered. On the other hand, many students also recognize that the result of measurements is not sufficient to show the generality/ validity of a statement(Kakihana & Shimizu, ibid.). It is important to investigate how students recognize and are convinced of the generality/validity from measurements, and how students shift measurements to conjecture and then to proof. The purpose of this research is to investigate the students' strategies when they solve proof problems, focusing on the use of measurement and on the shift to "proof". The implications for the development of a new method of teaching "proof" in the computer environment is also considered.

II Method

The activities of five pairs of either first or second grade women's junior college students are reported in this paper. Students had not studied geometry since junior high school and reported that they did not do well in geometry classes. They solved each proof problem for 20-30 minutes after learning to how to use (1)*Cabri-Geometry* for one hour. One computer was used by each pair. Students were given a problem on a worksheet and constructed figures by themselves on a screen. Videotapes, observation reports and written worksheets were analyzed.

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(1) Cabri Geometry was developed by Yves Baulac, Franck Bellemain and Jean M. Laborde at the LSD(IMAG), University Joseph Fourier, Grenoble, 1988. A Japanese version is available for NEC machines.

The following problem which is a basic problem in junior high schools was given to the subjects.

Problem

Construct △ABC and make midpoints M and N on each side AB and AC. What kind of geometrical properties can you find when you connect M and N? Then explain the reason why you can say this and write a proof for it.

III Results and Discussion

During their conversation, students moved points and observed the invariance of measurements on the figure. Fig. 1 is the result of analysis of their conversations and worksheets.

Fig. 1 The results of each group's activity

DY: When they moved the figures before measuring, each group noticed each item.

ME: When they measured the sides or angles,
each group noticed or were convinced of each item

RE: They started to make reasoning during conversations.

PR: They constructed formal proof (in writing).

group 1

Yumi&Chiharu	DY	ME	RE	PR
triangles are similar			○	○
Parallel			○	
bases are 2:1				
areas of triangles are 4:1				

group 2

Ryoko1 & Ryoko2	DY	ME	RE	PR
triangles are similar	○	○	○	○
Parallel	○	○	○	△
bases are 2:1		○	○	
areas of triangles are 4:1				

group 3

Emi & Sayuri	DY	ME	RE	PR
triangles are similar		○	○	△
Parallel		○	○	△
bases are 2:1				
areas of triangles are 4:1				

group 4

Emi & Sayuri	DY	ME	RE	PR
triangles are similar		○	○	△
Parallel		○	○	△
bases are 2:1		○	○	
areas of triangles are 4:1				

group 5

Sanae & Akiko	DY	ME	RE	PR
triangles are similar				
Parallel				
bases are 2:1				
areas of triangles are 4:1				

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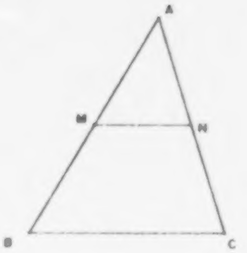
(1) Similarity of triangles

All pairs demonstrated that △AMN and △ABC were similar. In their conversation, they remembered the conditions of similarity of triangles as they moved and measured the figures. They also observed the properties, pointing out △AMN and △ABC and saying .

"This and this are similar, aren't they? "

Then they talked about their reasoning during their conversation.



Yumi "This length and this length are the same."
Chiharu "Because these points are midpoint."
Ryoko1 "Point M and N are midpoints, therefore 2:1"
Miyuki "It is half.",
Satoko "As a matter of fact, they are midpoints."
Miyuki "How can we prove it?" ,
Satoko " Because, C is the same and the ratio of the sides is 2:1."
Miyuki "It is not sufficient to show it by measurement, is it?"

In their writings of proof, four pairs wrote a more formal proof.

Miyuki and Satoko (Fig.2) wrote an example with measurements on their worksheet and wrote an insufficient proof.

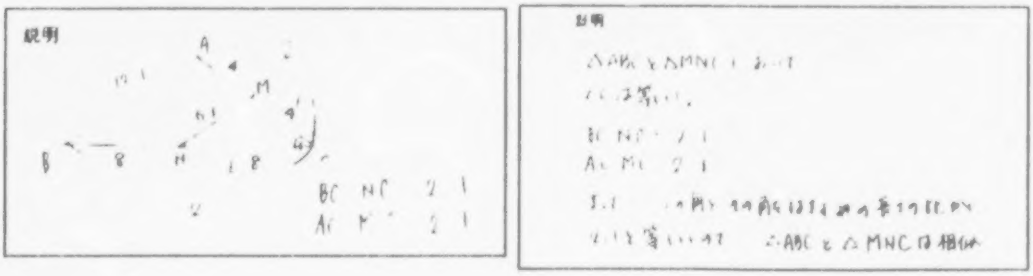


Fig 2 Miyuki & Satoko's explanation and proof

(2) Bases BC and MN are parallel.

In the protocols of four of the pairs' conversation, the following sentence is commonly found.

"These are parallel, because these angles are the same."

However, they did not write down the statement except for one pair (Yumi and Chiharu, Fig.3).

Fig 3 Yumi & Chiharu's proof

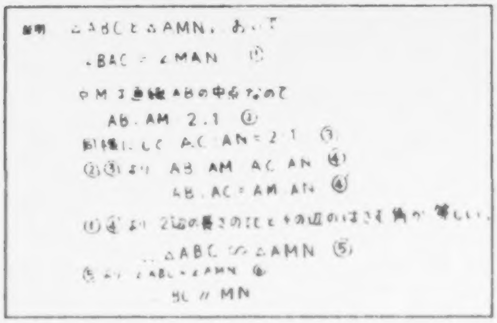
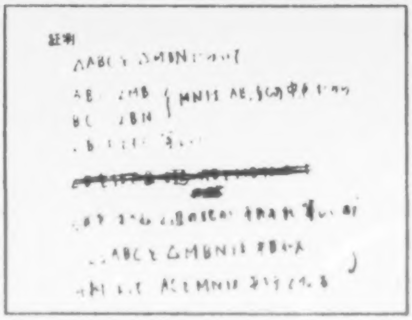


Fig 4 Ryoko1 & Ryoko2's proof



Yumi "When angles are measured, we do not need to prove, do we?"

Yumi "Measurements are not always the same. What should we do?"

Chiharu "We have to prove the similarity of these triangles.

One pair (Ryoko1 and Ryoko2) were convinced by measurements that the sides are parallel when corresponding angles are the same. However, they didn't describe it in their proof (Fig.4). Emi & Sayuri were also convinced by measurements that they did not feel the necessity to write a proof formally (Fig.5).

Sayuri "Because the measurements show it.

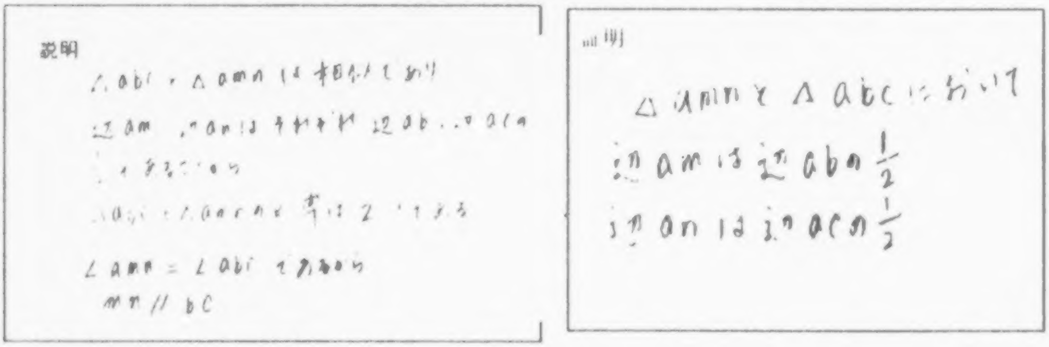
Emi " Explanation? Proof?"

Emi " In explanation, the triangles are similar , therefore MN is parallel to BC"

Sayuri " In the proof, I should write the conditions of similarity, shouldn't I?"

Emi "It is better to check the equality of angles by measurement. "

Fig 5 Emi & Sayuri's explanation and proof



Students know the statement that corresponding angles are the same when sides are parallel. Now, students understood its real meaning by measurement.

Ryoko1 "It is true that these angles are the same when sides are parallel. I've never checked it by myself. "

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(3) The ratio of base BC and MN are 2:1.

Ryoko1 "This is a half of this", pointing to the side MN and BC.

Ryoko2 "How can we find this is a half without measurement?"

They could not connect this statement to the conditions of similarity.

Sayuri "It's half" pointing to the side MN and BC. And move the point A.

"No, sometimes it's not half"

"How do I explain it?"

Emi "Because the triangles are similar."

"We have to write the conditions of similarity."

None of the pairs wrote a proof for this statement.

(4) The area of △AMN is one fourth of △ABC.

One pair (Sanae and Akiko) pointed out this fact. At the beginning of their activity, they thought the area of △AMN was a half of △ABC and then they moved the points and said "Is it one third?", "No, the same triangle is this part and this part. So it's one fourth!" At last, they concluded that the area of △AMN is one fourth of △ABC.

From these results, it was identified that the use of "continuous variation of geometric figures" and "measurements" in pair conversations helped to convince students of the characteristics of figures and theorems or axioms and then to help them to connect the inductive approach to the deductive proof. As typically seen in group 2, the use of measurements and dynamic transformation in computer environment provide an opportunity for students to reason why the conjecture is valid and general. All cases in which pairs were able to move and measure were successful to find a logical path of reasoning. Students do not understand that deductive proofs hold for all cases which satisfy the initial condition (Chazan,D.,1988). Even if students know that measurement is not sufficient to prove and that they have to write a deductive proof when they solve a geometrical proof problem, they were not convinced by these statements which they wrote in a proof by themselves. For students, proof means to write down a deductive reasoning process with mathematical symbols in a precise and standard manner. The use of "continuous variation of geometric figures" and "measurements" in pair conversations helped students to understand the necessity and the role of deductive proof. However, when using software, students usually do not write down on a worksheet any information which they found during conversations . Teachers need to instruct students to write down the things which they think and talk about when using software.

IV Conclusion

From measurement to conjecture observed in this case study, key factors were the use of measurement and dynamic transformation in computer environment. From conjecture to explanation or to proof, at the pair situation, the use of information previously obtained and the confirmation by measurement and movement of figures in computer environment seemed to help students provide a logical explanation and sometimes a logical proof. It was also found that students in this study did not distinguish the difference between explanation and mathematical proof. For an explanation in general sense, measurements will be enough base to show validity. But, for proof in mathematical sense, measurements is not enough to show validity. It is required to be shown by logical path. The view of explanation in general sense and that of mathematical proof that students have will play crucial role in computer environment.

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MATHEMATICS TEACHERS' TRAINING : SOME REMARKS ABOUT THE ROLE OF SELF-IDENTIFICATION

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Abstract

This paper presents the data of a research referring to the teachers of Mathematics after the end of their pre-service training. The research data show a clear difference between the change that the teachers have undergone during the training process and the way they themselves conceptualize this change. The paper attempts to interpret this phenomenon in the framework of the concept of self-identification, that plays an important role both in the way the trainees evaluate their training course and in the way they evaluate its influence upon them.

1. Theoretical Framework

Over the last years, research in the field of teacher education has become increasingly important to Mathematics Education. A large amount of articles about this subject indicates the diversity of the involved particular issues [ICME 7, Handbook, Houston, 1990].

One of the most important issues in the literature of teacher education is relevant to the conceptions and beliefs that mathematics teacher have about Mathematics and Educational Process [Tompson, 1986, 1992, Steinbring, 1988, 1991, Brousseau, Centeno, 1991, Koehler, Grouws, 1992, Arsac, Balacheff & Mante, 1992, Cooney, 1994, etc]. These conceptions influence their attitude towards both the training and the educational process. Relevant studies have shown that the teachers' conceptions and beliefs are deeply rooted in themselves, they are constant and can hardly change [Tomson, ibid., Steinbring, ibid., Arsac, Balacheff & Mante, ibid., Stigler, Perry, 1988, Lappan, 1992, Tzekaki, 1992, Kaldrimidou, Ikonomou, in press]. The findings make many researches attempt to find alternative ways for teacher education, focused on changing their conceptions and beliefs [Meredith, 1995, Bouffi, 1994, Bottino, Furinghetti 1994, Ferrandes 1994, Pehkonen, 1995 etc.].

All these approaches are realized within different theoretical frameworks such as social constructivism, problem-solving processes, group procedures, introspection, meta-cognitive discussion, etc., according to "the researchers' preferences and values", as it is indicated by Vinner [Vinner,1995]. However, these differences principally attest the absence of a theoretical framework appropriate to define the process of the development and changing in attitudes of the mathematics teachers, as far as their education is concerned. Many researchers share the same view and refer to the absence of an appropriate theory in their articles [see Meredith, ibid., Coney, ibid., Vinner, ibid.].

The research of a suitable framework for the concepts and tools that will contribute to the analysis and interpretation of such a complex phenomenon as teacher education and changing of their attitudes towards the education process, is not an easy work. Nevertheless, a close examination of the attitudes and the opinions of mathematics teachers orientates researches to the idea of "self-reference" and "self-identification" such as self-learning [Meredith, ibid.], frustration, awareness, credo [Vinner, ibid.], change of roles [Pehkonen, 1995] etc.

Thus, we believe that the idea of *self-identification* could play a very important role to the conceptualization, the treatment and the interpretation of phenomena relevant to the constancy of teachers' conceptions about educational practices.

It is commonly accepted that the interaction between teachers and their training environment results in a process of conceptualizing the issues presented during the training process, according to their cognitive system, past experiences, self-conception and self-identifying attitude toward training [Arsac etc, ibid, Brouseau, Centano, ibid, Fort, 1994]. Even if the trainees admit neither being eager to change their conceptions about teaching, nor being influenced by their training, they are eventually guided as to pinpoint and apprehend differently the phenomena taking place in their mathematics class and, thus, to change gradually their point of view and attitude [Douady, 1991]. Therefore, the teachers adopt new teaching methods even if they do not seem to clearly identify this modification.

The above mentioned consideration is further reinforced by the data analysis of a research on the conceptions of mathematics teachers about their training process in relation with their conceptions about the teaching of mathematics[1]. The first impression we get after evaluating the influence of the training operation confirms the constancy of the students conceptions, their difficulty in the changes and the assimilation of new theories, as well as their solid reaction against adopting them. But a closer examination of the influence of the training course alters our first impression. In other words, although the teachers themselves claim that they did not change their views on teaching or their way of considering some phenomena, in fact, they show a significant modification as far as choices of their teaching methods are concerned.

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[1] This research is conducted by M Kaldrimidou, A. Iconomou, P. Iconomou , M. Tzekaki.

2. Research Outline

The findings that follow result from a wider research[1], as it has already been mentioned, that was investigating the conceptions of mathematics teachers relevant to the necessity, the contents and evaluation of the training in relation with the conceptions about teaching. The research is carried out in two phases : one *before* and one *after* the completion of the training, aiming to detect possible changes in the conceptions under the influence of the training process.

From the data of this research, we analyze in this paper, only a few questions, those that specially concern the effect of the training in relation with the trainees' beliefs about this effect.

The above mentioned research was carried out in four Peripheral Training Centers (P.T.C.) from different Greek regions with questionnaires, answered by *66* mathematics teachers *before* and *after* the completion of their pre-service training. It has to be pointed out that the teachers' age ranges from 34 to 38 years old and they all have graduated from various Mathematics Departments 12 years ago. These teachers have spent the biggest part of these 12 years working outside institutional educational schemes (schools), in private organizations or delivering private lessons.

3. Main findings

We focus our attention on the questions giving, on the one hand, data concerning evaluating characteristics of the training and, on the other, data concerning the attitude of the trainees toward teaching.

α. Findings concerning evaluating characteristics of the training

• The usefulness of the training : The attendance at the training does not alter significantly the trainees' opinions about the usefulness of the training (see tab.1). Neither can we observe a significant change of the percentages, nor can we observe remarkable differentiation in the attitudes of each individual (before and after the course), as it is shown from the statistically significant correlation of the variables.

	before the course	**after** the course
1. It is essential	**22** (33.3%)	**21** (31.8%)
2. It is interesting	**35** (53.0%)	**35** (53.0%)
3. It is a waste of time	**09** (13.6%)	**10** (15.2%)

table 1. Training usefulness

• The expected training model The suggestions of the trainees relevant to the expected training model remains correspondingly unchanged *before* and *after* the course. The search for the "correct way" of teaching, either theoretically or with exemplary teaching, is obvious in the trainees' answers and expected by a large amount of them. This conception does not change at all, as it is shown from the correlation of variables. This finding is indicative of the teachers' conceptions, because the training did not intend to provide them with the so called "correct way" of teaching. On the contrary, their training was focused on Mathematics Education, that is on the theoretical and practical interpretation of the phenomena appearing in mathematics classrooms and the development of concepts and tools suitable to help teachers encounter these phenomena .

	before the course	**after** the course
1. **Presentation of particular teaching methods by the trainers**	**20** (32.3%)	**19** (32.3%)
2. **Attendance and analysis of teaching realized by the trainers**	**34** (54.8%)	**31** (52.5%)
3. **Persecution of theoretical concepts and design of teaching approaches by the trainees.**	**08** (12.9%)	**09** (15.3%)

table 2. Training model

• Evaluation of the training. After evaluating their training, 55.4% of the trainees claim that their attendance at the training Center resulted in improving their teaching efficiency, while the remainder 44.6% cannot think of a positive contribution of their training, as far as their teaching ability is concerned.

c. Findings concerning attitude characteristics of the trainees toward teaching

• Teaching efficiency. The teachers, *before* the course, characterize themselves, to a significant percentage (37.9%), as "efficient", as far as teaching is concerned, while the biggest amount (48.5%) claim that they have "some deficiencies". The percentage that claims "serious deficiencies" is very limited. In other words, the teachers up to 86.4% consider themselves as "efficient". The attendance in the training Centers results in reducing the percentage of the teachers who considered themselves to be "efficient" (31.8%), but it also results in reducing the percentage of those who regarded themselves having "serious deficiencies" (9.1%) with a corresponding increase of those who regarded themselves having "some deficiencies" (59.1%). The main tendency of these conceptions, *before* and *after* the course, is statistically very significant. Therefore, even if we can detect a minor change in this issue, the trainees do not seem to consider, in general, that the training course helped them considerably to improve themselves.

- Teaching methods. The examination of the trainees' choices in issues relevant to teaching methods and students' errors allow us to pinpoint modifications, due to the training course. These modifications show that there is a differentiation between the effect of the training process and what they themselves conceptualize as effect.

As far as the teaching methods are concerned, *before* the course, many trainees (32,3%) focus on solving exercises supported by mathematical theory, while most of them (38.7%) adopt the so called "traditional" model where the theory is presented first and then exercises are solved so that the theory is consolidated. Several teachers (29%) adopt the model of "solving problems" which are solved by the teacher himself. Finally, no teacher (0%) adopt a constructivism direction.

After the training course, there are several considerable changes in this issue. The "solving exercises" model diminishes to 13.1% and the "traditional" model to 29.5%. There is an important increase of those who adopt the model of "solving problems by the teacher" (49.2%), while a percentage of 8.2% already adopts a constructivism direction. In this phase, the trainees are asked to choose the teaching method they would adopt "under better conditions", that is better working circumstances and better educational system. Thus, "under better conditions" no teacher (0%) chooses the "solving exercises" model while the constructivism direction is adopted by almost half of them (50.8 %).

- Attitudes towards students' errors. The majority of teachers (60%), *after* the completion of the training course, continues to attribute the cause of the students' errors to their lack of knowledge, although, one third of them attributes the responsibility for these errors to the educational system and the insufficiency of teaching. The correlation of variables indicates that the 64.3% of the trainees answer the question keeping a traditional attitude towards errors (the students do not know), while 35.7% of them adopt a more improved attitude about errors. Finally, 23.9% of the trainees attempt to follow a more modern theoretical framework (see tab.3).

1. I repeat, remind	18 (39.1%)
2. I explain, use examples	17 (37.0%)
3. I give specific problems	11 (23.9%)

table 3. Encountering students' errors

On the whole, the study of the factors of the above mentioned data, indicates the formation of two main tendencies, relevant to the existing professional experience of the teachers: one implies a modern teaching model followed by a similar attitude towards students' errors, while the other one remains traditional with the same attitude towards error encountering.

4. Discussion

Summarizing the following points could be presented:

- The trainees had initially a neutral attitude toward the training course which, to some extent, they maintained up to the completion of the course, doubting the benefits they gained. They also maintained the same conception about the expected training model and its necessity.
- This attitude is relevant to self-identifying characteristics (self-evaluation of teaching efficiency, improvement) and not objective ones (contents of training course, methods etc.). As far as these characteristics are concerned they admit only minor improvements.
- The teachers' conception, however, about the best teaching method that focuses on the two basic models, is significantly differentiated: only one in three trainees maintains his initial conception while the great majority of them change this conception towards a positive direction. This change is even more obvious when the question is set in the framework of "better conditions", where 85.7% adopt the theoretical issues and methods they were taught during the training course. This assumption is partly confirmed by their views on responsibility and the ways they face errors.

The above mentioned data indicate a significant discrepancy between the teachers' evaluation on the outcome of the training course and the influence that they appear to have undergone [Fort, Tzekaki, 1994]. This remarkable discrepancy can also be related with characteristics of self-identification such as self-esteem, self-evaluation, self-conception about their role in the framework of the educational system [Maturana, Varela, 1988]; fact that is clearly implied, not only in the answers of the questionnaire but also in several conversations we had with the trainees and in certain observations of their teaching after the completion of the course.

Are the above mentioned results phenomenological and predictable? It is apparent that the trainees will initially change some components of their attitudes and later their beliefs and conceptions, that is their self-identifying attitude. In this sense, the idea of self-identification could be a basic key-concept for the interpretation of the complexity of the issues involved in training, contributing this way to the formation of a framework, appropriate for the study and the apprehension of the teachers' education phenomena. The concept of self-identification seems to be able to interpret both the trainees' reactions and their unwillingness to change their conceptions as well as the discrepancies between the influence they have undergone and the influence they acknowledge.

5. Epilogue

It has to be stressed that if we are to evaluate the way a training course influences the trainees we need to plan a long term research. There has to be some interval between the actual training and its evaluation, so that the extent to which the trainees were influenced can also be evaluated. Time is also needed so as to examine to what extent the difficulty of putting theory into practice, daily class routines, is true and not simply a result of the trainees' self-conception.

We would also, at this point, like to mention past experiences with training courses that have issued findings, not closely examined yet. These findings are indicative of the way mathematics teachers are influenced [Tzekaki, ibid]:

- Initially the trainees react in a negative way or at least in a neutral way towards training, having reservations about the theoretical concepts and tools presented during the courses. Nevertheless, they have been exposed to ideas that change their way of viewing teaching phenomena and eventually their own attitude toward teaching. These new practices share the same characteristics with those presented during the training process.
- The significant influences the teachers gained during the training might gradually fade away unless the teachers receive some kind of feed back; because as times goes by, the factors that have initiated their old teaching practices (teaching conceptions, particularities of mathematics, timing etc.) could re-appear.

The above mentioned phenomenon, relevant to the idea of self-identification, might contribute to formatting procedures for mathematics teachers' education.

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TO HAVE OR NOT TO HAVE MATHEMATICAL ABILITY, AND WHAT IS THE QUESTION

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This paper offers an example of evaluating mathematical talent, using a unique question as a source. 268 answers to such a question, given by students in grades 9 and 10 during a mathematics selection test, were cognitively analyzed. Classification was made based on the quality of thinking revealed in the answers, and a qualitative scale was constructed. The scale was statistically compared with the psychometric scale by which the test was initially evaluated and with other data. High correlation was found between the qualitative scale and the psychometric scale in questions associated with mathematical thinking. Low correlation or none at all was found in questions associated with learnt mathematical knowledge. It is therefore suggested that evaluation by means of cognitive analysis could be helpful in locating people whose mathematical talent does not always follow the conventions of the mainstream.

Mathematical ability means not only mastering acquired mathematical skills, but also expressing qualities such as originality, creativity, clarity and elegance, while solving mathematical problems. Obviously, exposing such qualities is possible only if the problem presented is not one familiar to the subject. If a person has solved a few problems of the same type in the past, he most likely develops a schema for solving this type of problems (Mayer, 1982). Schoenfeld (1982) claims, that operating a well-established schema in a successful manner is still no evidence that real mathematical thinking took place. It follows that conventional tests, including similar questions to those that have been previously exercised, are inadequate for evaluating mathematical ability in the wider sence. Such tests can be used to check the effectiveness of learning on the one hand, and as a predictor of future success in the educational system on the other hand. However, in order to evaluate mathematical talent, it is advantageous to use *unusual* questions.

Schoenfeld (1985) and Krutetskii (1976) formulated many such questions for their research. These works illuminated an important point: The level of mathematical ability is determined not only by the correctness of the answer, but to a large degree by the quality of thinking revealed while the subject is solving the problem. An example from Krutetskii (1976) illustrates this well. Consider the following question:

Three pupils visit a library on different days. One comes every third day, another every fourth and the third every fifth day. The last time they met in the library was this Tuesday. How long will it take until they meet there again, and on what day will that be?

A seventh grade student solved this problem by writing down consecutive numbers starting at 1 and marking every third, forth, and fifth number with different marks. The marks coincided on the 60th day. He counted the days and discovered that the pupils would meet on Saturday, after 60 days. A second student said: "It's the least common multiple!". She calculated 3•4•5=60, divided 60 by 7, obtaining 8 and a remainder of 4, and declared: "Wednesday, Thursday, Friday, Saturday - 8 weeks from Saturday". Evaluation of these two solutions by a psychometric criterion, i.e., correct/incorrect, is liable to present these two students as equally able. Yet, the processes used by the students are on completely different levels and reveal disparate qualities of mathematical thinking. This example strongly emphasizes the importance of cognitive analysis. The purpose of our study was therefore to examine what can be achieved by using unconventional questions and evaluating the solutions by cognitive analysis.

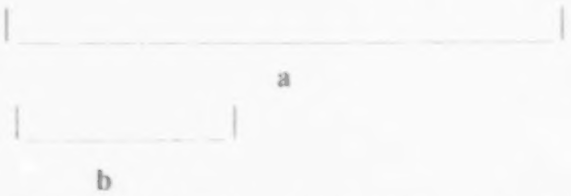
METHOD

Subjects and Procedure

The test used in this study was initially designed as part of a selection process for a unique school in Israel. It was intended to evaluate the mathematical abilities of the applicants, 268 students in grades 9 and 10 from all over the country[1].

After a task-analysis was carried out for each of the 18 questions, we selected 3 questions as being unconventional and therefore appropriate for our purpose. Cognitive and statistical analysis were performed for each of these questions. Due to space limitations, we shall here review the cognitive analysis (and later the statistical one) for one question only. We termed it "the estimation problem":

Rachel estimated the length of line a and Sarah estimated the length of line b.



After measuring the lengths of these two lines, it turned out that the difference between Rachel's estimation and the exact length was 2 cm, and the difference in Sarah's case was 1 cm.

Choose the correct statement and explain:

(1) Rachel's estimation is better because

(2) Sarah's estimation is better because

(3) Both estimations are equally good because

(4) The merit of the estimations cannot be compared because

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[1] The test was composed and administered by the Szold Institute in collaboration with an expert committee led by Dr. Nurit Zehavi of the Weizmann Institute.

Task-Analysis of the Estimation Problem

In this problem, the student has to deal with the notion of estimation's merit. Merit of estimation might appear in everyday discourse. As such it does not have a well defined meaning. However, the notion does involve an accurate technical meaning. Namely, in a better estimation the ratio between the error and the real magnitude is smaller. When asking the student to choose among the different statements, we want to know this: Is the student aware of this technical meaning? If not previously known, can he or she *construct* this meaning? If we assume that this type of question is unfamiliar to the student, then actually he or she is required to accurately elucidate a term of general or even obscure meaning in everyday language. The student must perform a conceptual analysis that is beyond the immediate given data. To successfully deal with the task, the student must perform the following steps: Examine the two lines and notice that the length of **a** is more than twice the length of **b**; Compare the errors in the estimations of Rachel and Sarah, relative to the length of the line estimated (e.g., compare the error of 2 cm in line **a** to the error of 1 cm in line **b**). Considering the ratio between the lengths of the two lines, he should then conclude that Rachel's error is smaller and therefore her estimation is better.

Categories for the Estimation Problem

Answers were classified within the following nine categories (N=268). Each category is illustrated by one or two typical explanations.

(A) Rachel's estimation was chosen as best, because the ratio of the error to the length of the line was smaller in her case (37 students; 13.8%).

> *"Line **b** is smaller than half of line **a** and therefore Sarah's error would have been greater than 2 cm relative to line **a**"*

(B) The statement that the estimations are equally good was chosen because the student thought that line **a** was twice the length of line **b** and therefore the ratio of the error to the length of the line was the same (19 students; 7.1%).

> *"Line **a** is twice the length of line **b** (as I see it) and that's why they share the same error because 1 cm (of Sarah) X 2 (to get to **a**) will be equal to 2 cm, like Rachel's estimation."*

(C) The statement that the merit of the estimations cannot be compared was chosen because the ratio between the lines was unknown (17 students; 6.3%).

> *"It's not possible to know the ratio between the two lengths and therefore we don't know the percentage of Rachel's mistake and the percentage of Sarah's."*

(D) The statement that the merit of the estimations cannot be compared was chosen because the lengths of the lines were unknown (34 students; 12.7%).

> *"The lengths of the lines are unknown to us and it's not possible to calculate how much the estimation deviates relative to the length of the line."*

(E) The statement that the merit of the estimations cannot be compared was chosen because estimations can be compared only when the same line, or equal lines, are estimated (44 students; 16.5%).

"Each one estimated a line of different length so you can't compare between them!"

(F) In the student's answer, regardless of the statement chosen, the idea of the ratio between the estimation and the true length was mentioned, but only vaguely and unclearly (17 students; 6.3%).

"The merit of the estimations cannot be compared because the lengths themselves are not equal and we cannot know what is the percentage of each distance in each one's estimation."

(G) In the students answer, there is no attempt to accurately define the concept of "estimation's merit". The meaning attributed to the concept is as pedestrian as the common everyday meaning of the term "estimation" (14 students; 5.2%).

"Estimation is an inaccurate thing and so there is no such thing as a better or worse estimation - most are inaccurate."

"The merit of the estimations can not be compared because every person might estimate the same length differently"

(H) Sarahs' estimation was chosen as best because the difference between her estimation and the true length is smaller. In this case, there is an attempt to technically define the concept of "merit of estimation" (the difference between the estimation and the true length), but the logic of this definition is erroneous (54 students; 20.2%).

"The difference between her estimation and the true length was smaller, so she was more accurate than Rachel was."

(I) No answer or non-classifiable answer (32 students; 11.9%).

The Categories and Levels of Thinking

The intent to preserve a wide variety of answers, of different quality and nuances, led to the nine categories described above. This classification, however, is insufficient for ranking the students by the quality of their mathematical thinking, since it proposes no hierarchical order. The next step was therefore to determine the level of thinking characteristic of each category, unite categories of a similar level and create a graded scale. This process will now be described.

Dealing with the task, as we suggested, forces the student to offer a meaning to the concept of estimation's merit. We claim that a student reveals a higher quality of thinking, when the meaning he offers is closer to the technical definition previously cited. According to this, category (A) represents the highest quality. Here, Rachel's estimation was chosen as best because the ratio between the error and the length of the line was smaller in her case. Nevertheless, category (B) can be regarded as having the same quality. The student thinks that line a is twice the length of line b, and since this is also the ratio between the errors, the estimations are equally good. This student, although making an optic error, applied the same principle as a student choosing the first statement, and thus cognitively their answers are of equal value. These two categories were therefore ranked together as the highest level of thinking. The next category in terms of quality of thinking is category (C). Here the chosen statement is that the merit of the estimations cannot be compared, because the ratio between the lines is unknown. Hence, the student is aware of the dependence between estimation and ratio, and probably would have evaluated Rachel's estimation as better, had the ratio between the lines been given. Yet, this information is unnecessary; It is sufficient to observe that line **a** is more than twice longer than line **b.** This observation was not made by the student, who could not find ways to cope with the apparent lack of information. Category (D) is quite similar, except that "missing data" is the *length* of the lines. Actually, regarding the lengths as necessary information can be perceived as reflecting less quality of thinking than seeing the ratio as essential. However, since knowing the length of the lines could lead to applying the ratio principle (see example in (D)), we ranked these two categories together as the second level of thinking.

Let us now examine category (E) (Estimations can be compared only when equal lines are estimated) and category (F) (Vague concepts). Although different from one another, both categories convey a feeling that the situation described in the question "annoys" the student. He does not feel confident. Had the same line been estimated, he would have had no problem - the closest estimate would have been chosen as best. The fact that two different lines were given confuses him. Students assigned to (E) settled this confusion by simply not dealing with it; They claimed that the comparison just can not be made. For students assigned to (F) the confusion was also expressed by indiscriminate use of terms, such as "percent", "accuracy", "difference", etc. In all these students the concept of estimation's merit is not sufficiently developed. This also applies to category (G). Here too, the concept of estimation's merit of is not developed and is given an everyday meaning, while the technical meaning is completely absent.

Finally, category (H) is left. In this category, the student evaluates the estimation by the absolute error, that is, by the *difference* between the estimation and the correct value instead of by the *ratio* between them. This reflects a famous misconception known as "the additive approach" (Karplus and Peterson, 1970). Karplus and his associates recognized this misconception in their work on proportional reasoning (Karplus et al., 1983). They found that a certain percentage of the population examined used additive approach when solving problems requiring the use of ratio. It is interesting to note that this percentage is reported to be decreasing when the problem involves numbers from different dimensions - for example number of candies and cents - as compared to problems involving only one dimension, such as length. In the estimation problem discussed here, there is only one dimension - length - and our statistical results will be shown to be in agreement with the findings of Karplus and associates. It is difficult to rank this category in comparison to the previous three that were defined as undeveloped conceptions of estimation's merit. This is a misconception, not an undeveloped conception, but it can also be associated with temporal cognitive underdevelopment, since it is typical of younger children: Inhelder and Piaget (1958) regarded the ability to use ratio as an essential component of formal reasoning, developing with age. Thus it is difficult to determine which conception is of lower quality, and so we decided to rank them together at the last level of thinking. (However, the additive misconception was separated in the answers distribution so that the percentage of the students holding it could be compared to Karplus' findings).

RESULTS

As described above, a qualitative scale was constructed, consisting of three main categories and an additional category of students who did not answer the question or gave non-classifiable answers (Some of these answers will be discussed at the end of the paper). Table 1 presents the distribution of the students according these categories.

Category	Short description of the category	Number of students	Percentage out of total students	Percentage out of classified answers only
1	Correct perception of the notion of estimation	56	20.9%	24%
2	Connecting estimation to ratio or length but avoiding from application	51	19%	21%
3 (a)	Undeveloped perceptions of the notion of estimation	75	28%	32%
3 (b)	Additive thinking	54	20.2%	23%
	Total classified answers	236	88%	100%
4	Non-classifiable answers	32	12%	-

Table 1. Distribution of the answers to the estimation problem (N=268)

It is noteworthy that the additive thinking category contains 23% of the classifiable answers. This finding is very similar to that of Karplus and Peterson (1970) who found that out of 75 students of grades eight to ten, 25% belonged to this category.

Comparison of Evaluating Students by the Qualitative Scale and by a Psychometric Scale - Some Statistical Analysis

The psychometric scale we refer to is the one used by the institute in which the test was originally designed and checked. It does not relate to qualitative thinking, but rather to the existence of certain features in the answer to which a numerical value was assessed according to a well-defined key. In the estimation problem as evaluated by the institute, a maximum score of 2 points was possible: one point for selecting the right statement (the first one) and one point for an explanation which included one of the terms "ratio", "error percentage", "relative error" or a similar term.

When we compared the distributions of the answers in the estimation problem according to both scales - the qualitative one and the psychometric one - we found that there was a fairly good correspondence between them (the detailed analysis will be given elsewhere). This is a very reasonable outcome; It is well known, for instance, that the intelligence tests constructed by Binet and Simon are highly correlated with Piaget's tasks (see Terman & Merril, 1960). The assumption that a psychometric measurement could correspond with a cognitive measurement has therefore evidence to support it. Yet it is interesting to confront this finding with the following findings, concerning the psychometric score in the rest of the test, and school achievements.

The mathematics test consisted of two parts, designed to examine the mathematical *knowledge* and the mathematical *thinking* of the student, respectively. We used the psychometric scores of the subjects in the test for further statistical analysis. The findings were as follows. The correlation between the qualitative scale constructed in the estimation problem and the score in part I of the test was low and not significant for all students. However, the correlation between the qualitative scale and the score in part II was fairly high and significant ($r = 0.37$, $p < 0.01$ for ninth graders, $r = 0.53$, $p < 0.01$ for tenth graders). The correlation between the qualitative scale and the *total* score of the test (parts I+II) was 0.19 ($p = 0.017$) for ninth graders and 0.43 ($p < 0.01$) for tenth grade students.

Another correlation that was calculated referred to the mathematics school mark of those students who were accepted to the school, at the end of their first year of study. The correlation between this mark and the qualitative scale in the estimation problem was low and not significant for both age groups.

DISCUSSION

The estimation problem was chosen for the research because it is unusual, and thus can serve as a potential source for revealing mathematical talent, as we claimed above. The low correlation between the qualitative scale based on this question and the score in the first part of the mathematics test is therefore not surprising, since this section of the test was meant to check acquired mathematical knowledge in a rather standard manner. On the other hand, the correlation with the second part of the test is relatively high, as can be expected. Indeed the test constructors viewed this section as a mean of evaluating mathematical thinking which is beyond learnt algorithms. The lower correlation with the total score of the test can be explained by the fact that knowledge questions had considerable weight, but there is another interpretation that is noteworthy. The selection test was designed to predict future success in the school system, and as such it is closely related to the main stream in mathematical education. Performance of high quality of mathematical thinking in a unique question does not necessarily foretell high achievement within the current framework of mathematics education. The fact that our qualitative scale did not correlate with mathematical achievements in school supports this interpretation. The common school system lacks appropriate tools to appreciate unconventional thinking. We realize that apparently not many students fall into this category. Nevertheless, it is an issue worth thinking about. For instance, consider the following phenomenon. A few students (less than 2%, and therefore omitted from classification) marked more than one statement in their answer (generally the whole four of them) giving each statement an explanation, although the statements clearly contradict each other! One of these answers, for example, was:

Rachel's estimation is better, because looking at the percentage and relative to the length of the line her estimation is more accurate. The ratio between the difference and the length is better.

Sarah's estimation is better, because considering numbers she only missed by 1 cm, and that is closer to reality than Rachel.

Both estimations are equally good because in both cases the difference wasn't too large.

The merit of the estimations cannot be compared because we don't know the lengths of the lines, and so we can't calculate the ratio between their answers and the true length"

Why did the student write such an answer? He does not appear to be hesitating, on the contrary; he confidently supplies a rather convincing explanation to each of the statements. Perhaps he just hoped to gain more points. Yet, it seems like the student had challenged himself to succeed in finding reason in each of the statements. We believe that this student is not inferior to his peers who chose the right statement. He just has a more unique, even creative perception. The psychometric measure, however, rejected all these answers. We see this as an example of how a psychometric scale can fail to discover mathematical talent, whose original and creative side is, in our opinion, very important. In order to confirm our impression about these students, we examined their answers to other questions in the test. Indeed we found some beautiful answers.

In an educational system that uses only psychometric instruments and standard tests, the focus seem to shift from the issue of having or not having mathematical ability to the issue of having or not having memorizing ability. It is not surprising that some competent, non-conforming students who think differently are somewhat lost. The system might discover these students if from time to time evaluations would be based on cognitive analysis such as the one discussed in this paper. If the questions given are unique enough, such analysis could be surprisingly beneficial.

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INVISIBLE ANGLES AND VISIBLE PARALLELS WHICH BRING DECONSTRUCTION TO GEOMETRY

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Geometrical thinking of important subjects of mathematics education, including: top-ability high school students, experienced mathematics teachers and professional mathematicians was investigated. The study revealed their common difficulties in applying the notions of angle and parallelism in space. Search through historical sources, textbooks end didactic literature had displayed, how meanings of these notions are blended with variable positions, taken for their teaching The result of the study is a critical reanalysis of current approaches to school geometry.

1. INTRODUCTION

This study continues the work on geometrical thinking (Kopelman & Vinner, 1994) of rather underresearched populations: top-ability high school students, experienced mathematics teachers and professional mathematicians. What is common to this populations is their certain success with learning mathematics. It is reasonable to ask: why to research them at all? The answer is, that, first of all, they are important participants in mathematics education, influencing it no less, or even more, than others. Second, since memory, motivation, knowledge, ability to deal with symbols, abstract reasoning, visualization and intuition are not lacking in that case, predispositions in their mathematical knowledge and sudden failures (not lacking either) speak of issues, which testing of usually sampled populations doesn't reach. To meet the challenge, research methodology should also be ready for change: if in a usual cognitive study meaning of tested notion is taken as pregiven, here it opens to criticism together with teaching of that notion - both viewed in historico-epistemological perspective. In other words, what gets into focus is not cognition grappling with ready-made mathematical notion, but variable context of the notion itself.

The viewpoint adopted by this, basically empirical, study has been influenced by the following issues raised in previous theoretical and historical works: of multiple subjectivities in mathematics (Rotman, 1994), orality/writing opposition in mathematical discourses (Otte & Seeger, 1994), role of teaching in reshaping pure mathematics (Grabiner, 1974) and historical dimension of mathematical knowledge, unveiled by Lakatos (Lakatos, 1976).

The study began from evaluation of ability to apply vectors by senior high school students at the end of their advanced course in three-dimensional geometry. Those of the asked questions to which students had given wrong answers, had been suggested to experienced mathematics teachers and mathematicians - most of whom, gave just the same answers as the students. A search for original meaning of problematic notions was undertaken. Reading of historical sources didn't reveal any instances when these notions could be isolated as carriers of some autonomous sense, but rather always blended with positions taken for their teaching. Brining this positions to light, resulted in the following critical reanalysis of current practices and didactics of teaching spatial geometry.

2. THE CASE OF ANGLE

The following question was suggested in a written test to 167 successful 12th graders chosen from four different schools in Jerusalem who studied advanced course in spatial geometry in classes led by highly professional and devoted teachers.

Question 1. Given a line in space and a point outside the line, how many planes in space may be drawn through this point which make an angle of thirty degrees with the given line?

The same question was suggested in different groups of 37 senior high school mathematics teachers at the end of their in-service training course in three-dimensional geometry. Observation of those who worked on the question had shown, that nobody felt need in calculations, relying upon imagination, or sometimes, helping themselves by drawing, or hands and sheets of paper. Answering this question, almost all of the respondents, students and teachers alike, were wrong about only two possible planes. Though according to the written test they were about 70 percent of the sample, oral interviews proved that most of those who answered differently used erroneous argument. Many assumed, for instance, that if to turn one plane which satisfied the condition around an axis, containing the given point and a point where the plane meets the line, this would preserve the angle. Those single ones who succeeded explained that first they had imagined the described turn of the plane, but then, in order to compensate decrease of the angle, directed the plane closer to projection of the given point upon the line. The question turned out to be difficult also to professors of mathematics. Among 12 professors who volunteered to be interviewed, 4 answered wrongly (among them two who had done it as a home task). Some might think of the question as a pitfall that only looks simple but in fact require or quite a formal calculation, or the shown ingenious kinaesthetic solution. But it's not: enough to imagine direction vectors of the line and possible planes as, in fact, was intended by the test.

But there is another quite striking aspect in this story: *nobody* - the narrator including - had started from relational nature of angle in that case. To explain what is meant by this, let us think about the following question: given a point in space, how many planes may be drawn through this point which make an angle of thirty degrees with a vertical? The answer is obvious: infinitely many. And it is really not important whether a vertical goes through the given point or not. A certain plane makes the same angle with each of verticals. Or, in other words an angle between a plane and a line does not change with parallel displacement of each of them, since it does not change their directions in relation to each other. All this is true for the initial question as well. If the relational nature of angle may be so important, what treatment does it get in geometry instruction? With introduction of angles in plane geometry students sometimes are warned that magnitude of angle does not depend on length of its sides. But do we ever suggest to students to find an angle between two drawn line segments which unfortunately intersect outside the drawing to show them that it does not matter for an angle, where both his sides meet, but only their directions? In solid geometry we, following the same tradition, define an angle between a line and a plane as the plane angle between that line and its projection on the plane - even we know well that for the angle between the line and the plane it does not matter, whether the mentioned plane angle will be formed at all: in perpendicular and parallel cases. Nevertheless, there is a moment in the course of teaching solid geometry, when the relational nature of angle between lines necessarily steps in, because the lines in this case do not intersect: in the notion of angle between skew lines. The definition of the notion should lead students, so it seems, to the relational character of angle and this was intended to test in the next the question:

Question 2. Given a line in space and a point outside the line, how many lines in space may be drawn through this point which make an angle of thirty degrees with the given line?

Only 18 of 134 students from the abovementioned sample and similar proportion of the interviewed teachers, while answering, had considered angles between skew lines and came to an infinite number of the possibilities. Since the question was really 'around' the proper definition, and the students were the best, one may only wish to have, it prompted to turn to circumstances in the history of geometry, which led to this, poorly picked up, definition.

The notion of angle between skew lines is relatively new in geometry. In 1748 Euler in his *Introductio* had demonstrated calculation of certain cross-sections, where he introduced as a parameter an angle between certain line segment in space and a line which he had drawn from an end of the segment parallel to one of coordinate axes. In 1773 another of the founders of analytic geometry, Lagrange, provides us with another evidence (Boyer, 1956)

> *...together with two angles p and q which determine the position of this radii, of which p is that angle which the radius makes with one of the axes, such as that with the z-axis, or rather with an axis parallel to this, but passing through the center of the radii, and of which the other q is the projection of the radius r on the plane of the x,y*

coordinates makes with the x-axis, or, which is the same thing, with an axis parallel to the latter and passing through the center of the radii...

In this extract, devoted to really new, for that moment, thing - polar coordinates in space - Lagrange explains the meaning of the two angles p and q, which, in fact, are angles between skew lines, in such a way, that enables the reader or to follow it literally, as exact details of his construction, or to see there deliberate stress on important general feature: angle as an immanent relation between two lines in space independent of their possible intersection. Fifteen years later, Lagrange, when it comes to formulae for projections of force on axes in his *Analytic Mechanics*, is still faithful to his description of this kind of angles. At the end of the 18th century rapid development of higher education and science in France had required to communicate the results of Euler, Lagrange and Monge in college courses. This caused substantial formative efforts from the authors of such courses, applied to form and content of the material, in order to deliver their course as effectively as possible. Thus, the core theorem of the course about projection of a line segment on an axis had accepted appealing formulaic form, but had to be preceded by a new definition of angle as extension of the older one, in order to include the case of a segment, which is skew to a line. Copied in the host of textbooks on analytic geometry, which appeared at that time the definition looked very different the exposition of the same thing by Lagrange: '*in case of nonintersecting lines, an angle between them will be adopted as the angle between two lines which are drawn from one point and which are parallel to the original lines*'. This definition, intended to secure the following it formulae, sounded more like a decree of a new angle, rather than revelation about its relational character; the latter evidently remained unknown at that time - otherwise the inclusion of this definition wouldn't be necessary. Finally, since the middle of the last century this definition began to appear in school textbooks (The Elements of Euclid ed. by I. Todhunter 1862/1933, p.290), but apart from analytic geometry in space. Judging from our times, we may claim that this didactical displacement of the definition had taken the corresponding notion away from its original context and thus has blocked for students any way meaningfully to apply it. The school didactics, satisfied with mere presence of the definition inherited from respectable source, has never started to "naturalize" it, in the sense of making relational nature of angle usual and obvious - like naturalizing negative numbers, for example.

3. THE CASE OF PARALLELS

Next notion, which turned out to be problematic, was the notion of parallel lines in space. This notion is taught in school geometry in two versions: in the traditional, following the pattern of Euclid's Elements, and in a more recent one, which has been initiated in school since the middle of our century and uses vector approach. The teaching of the latter one, which is the object of our study, heavily draws on students' intuitions about directions and parallelism in space. Contrary to the previously discussed angles in space, in this case there were discernible efforts to naturalize the notion of parallelism even on the level of didactical literature for mathematics educators themselves. Thus Hans Freudenthal suggested (Freudenthal, 1983), that parallelism - as equality of directions - is a mental object, which is imposed on us from early childhood by natural environment and products: roads, gates, rows of houses, edges of a ruler, of a sheet, of a box, etc. So, for students the vector approach to geometry should be even more direct than the classical one:

> *...Equality of direction of directed straight lines is an intuitively primary phenomenon. The discrepancy of direction of intersecting as well as skew line pairs is a striking phenomenon. The fact that there is a plane through parallel, though not through skew, lines is comparatively secondary and not immediately obvious. --- There may be reasons to choose this property to define parallelism of lines in a* logical *system of geometry, but they are not at all compelling. One can equally well imagine a system in which direction or equality of direction is one of the fundamental concepts.* (Freudenthal, 1983, p.305)

While there is no argument about the second part of the passage, the first one is still short of evidence. The point here is not that real things will never reach mathematical ideal. But what exactly is referred in them as example of parallelism is *intention* in their production towards that ideal, which answers the things' functional use. If to look again at the list of things, which as Freudenthal suggests, form imposed on us the mental object of parallelism, we may note that sides of a road should not necessarily be parallel: a road may curve, but its constant width is a dominant feature. Again, edges of a ruler should better be in one plane, but we really care only about straightness of one of them. So, the features, intended towards parallelism in the listed things, are hardly identical. At the same time, the impression of parallelism is absent if there is no build-in intention towards it.

Look at the following picture and imagine that these are two rays of searchlights in the sky. Nobody, who knows, what a searchlight means, will consider these rays as parallels, even if they are in a fixed position. Since it is very probable, that one of them is directed towards the viewer and the second - just the opposite way.

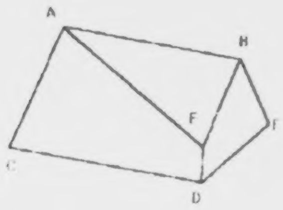
If we turn now to textbooks, we shall see that the notion of parallelism is linked here to notion of vector in way, which may be perceived differently, depending on a time and goal of instruction, as well as on a reader's viewpoint. According to well established textbook tradition (Wilson, 1901/1943), vectors are introduced as quantities, characterized by direction and magnitude, implying that equal directions in space are related to parallel lines. When this introduction was coined by mathematical physicist J. Willard Gibbs in his Yale lectures in 1881, he named it a definition, meaning it as a such for his students. From to-day mathematician's view it is not, because still leaves vector as undefined object until operations with vectors are specified. Or, according to accepted now in mathematics style, it must be supplemented by the promise, that "later the terms 'direction' and 'parallel' will accept exact meaning". This 'looping', or completeness, which requires later definition of initially intuitive terms, is important feature of modern mathematical rhetoric and is present even in school textbooks - but hardly may be appreciated by noninitiated learners. From quite a different point of view of a didactician, the introduction of a new notion of vector alluded to the special, two-dimensional case, drawing on the knowledge of previously studied plane geometry. Having this example, it is easier for a learner to make the main, most difficult step: to generalize it to the three-dimensional space. Since both the didactician's and the mathematician's intentions are far from being on the surface, a learner may still understand literally the above-mentioned introduction, which presents vector as it is, in all its totality, like most of the things, which are presented to him or her in, essentially oral, classroom. The soundness of the new notion stems from other notions, like the notion of direction, whose soundness, in turn, rests on common usage of the word and *agreement* about our ability to discern arbitrary directions in space. At the later stage of the course, after main operations with vectors are introduced, the notion of parallel line is defined: '*if one vector is a non-zero scalar multiple of another vector, then two vectors define directions of two parallel lines in space or relate to the same line*'. Usually, teachers illustrate this abstract definition with a simple drawing of parallel lines, marking their direction vectors. Again, within the pure mathematical perspective students got a definition of a *new* relation, thus completing the formal 'master plan'. Didactically, the defined relation was illustrated by the familiar, two-dimensional case. Within the third, 'literal' perspective, the complex of definition and illustration had described in vector language *the known* spatial feature and the illustration again referred to ability *at a glance* to recognize equality of direction in space. The latter notion is so common that may be found far outside one discipline. For example, in strong consonance with the quoted opinion of Hans Freudenthal, two well known psychologists write in their book "Language and Perception":

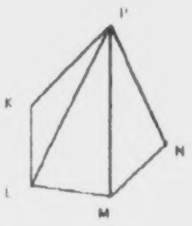
> *...Although the notion that faces or edges are parallel is not easily defined without more geometrical terminology than we wish to introduce, the perceptual impression is sufficiently direct and immediate...*
>
> (G.A. Miller & P.N. Johnson-Laird, 1976, p.53)

Still it is possible to ask: *is* it?

The following question was suggested to 113 of 12th graders and 12 teachers from the above-mentioned sample :

Question 3: There are two drawings of solids (polyhedra) before you. Find out and mark on each drawing all the edges which may be parallel to each other in space. (Note: an edge is a border between two adjacent faces which are not in one plane).





The results indicated that perception of parallel and skew lines may not be immediate. 88% of the students and all the teachers marked pairs AB and CD on the left figure as probably parallel (the wrong answer, according to either Euclid's or vector approaches). The students' method was to imagine the directions of the edges or to try to direct certain edges to be parallel. There is no doubt that the mistaken students didn't perceive the pictures two-dimensionally, since 94% of them didn't mark edges KL and PM as parallel. It also means that perception of parallel and skew lines is not a primary phenomenon, since both pairs of edges look graphically as the same "Z" pattern; they were approached differently, only because the right picture was associated with a pyramid. At the same time this association precluded 93% of the sample to consider the edges KP and MN as probably parallel.

The claims, that perception of equality of directions in space is immediate and the notion of parallelism is natural, can not make it more natural , than it is, and substitute educational efforts to naturalize it. All this doesn't say that the students do not know what parallel lines are. Being asked about that in the same test, they answered unanimously: parallel lines in space are the lines which have equal directions or, whose vectors of direction are proportional. In applications they easily deal with the parallelism, using intuition. But the looping, designed in their textbooks and intended to supply learners also with a formal outlook on that notion, has remained a literary move unrecognised by a mass reader.

4. CONCLUSION

The hard evidence, presented by the study, is the failure to solve correctly some qualitive problems in elementary geometry by advanced 12th graders, experienced mathematics teachers and professional mathematicians. It is not an educational disaster, neither a pure accident. It is not the diaster, since all the subjects in the study are very successful in their mathematics studies and some even belong to mathematics profession. It is not an accident either, since advanced students were chosen from different schools, the teachers were suggested these problems at the end of the relevant in-service course and the mathematicians are active professionals, coming even from different countries. The findings address the issues, which careful reading of teaching and didactical texts may reveal: meanings of the taught - and then, tested - notions are blended with variable didactical positions, which are taken to teach them. These positions stem from the long-standing tendencies, discernible within mathematics education, like: innovation and relying on certain tradition and authorities, concern with precision of expression and nurture of intuition, and, finally, aspiration for effective teaching - but which, without critical eye, may easily turn again themselves.

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RESEARCH ON THE COMPLEMENTARITY OF INTUITION AND LOGICAL THINKING IN THE PROCESS OF UNDERSTANDING MATHEMATICS: AN EXAMINATION OF THE TWO-AXES PROCESS MODEL BY ANALYZING AN ELEMENTARY SCHOOL MATHEMATICS CLASS

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Abstract: *The purpose of this research is to demonstrate the complementarity of intuition and logical thinking in a process of understanding mathematics basing on two basic notions of mental model and reflective thinking. In this paper, we examine the validity of the so-called "two-axes process model", especially the horizontal axis consists of three learning stages by analyzing an elementary school mathematics class. Firstly, we identify some mental models of length which students have initially at the class and lead to a misjudgement or a mathematically incorrect anticipatory intuition. Secondarily, we observe how such intuition has been changed under the control of students' reflective thinking in a whole-class discussion. As a result of the protocol analysis of a class, the validity of the horizontal axis of the model is documented.*

INTRODUCTION

In Japan it is one of main objectives of school mathematics education to develop student's intuition and logical thinking. To realize this objective, many mathematics educators and researchers have made extensive efforts in various ways. However, we can not say that we have satisfactorily realized the expected result. In consideration of the existing state of things, we should capture the nature of students' thinking in the teaching and learning of mathematics.

Koyama (1988) made a theoretical study on the relationship between intuition and logical thinking from view points of both the history of mathematics development and the developmental mode of human thinking. He states, as a result of the study, that intuition and logical thinking are complementary and closely interrelated in human mathematical thinking. In other words, human thinking could developed productively and soundly only when intuition and logical thinking are in a harmonious and cooperative relation. Recognizing the such complementarity and the idea of objectification or explicitation in the van Hiele theory (van Hiele, 1958), Koyama (1992a) made clear what characteristics a model of students' understanding mathematics should have so as to be an useful and effective model in the teaching and learning of mathematics. The models of understanding mathematics presented in preceding papers are classified into two large categories, i.e. "aspect model" (cf. Skemp, 1982) and "process model" (cf. Pirie & Kieren, 1989). Focusing on the process model of understanding mathematics, we recognize that reflective thinking plays an important role to develop students' understanding, or to make their thinking progress from a certain level to a higher level of understanding. Koyama (1992b, 1993) has explored basic components of students' understanding mathematics and presented the so-called "two-axes process model" of understanding as a theoretical framework for the teaching and learning of mathematics. The model consists of two axes in which the vertical axis implies some levels of understanding and the horizontal

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axis implies three learning stages at each level, i.e. intuitive, reflective, and analytic stage.

PURPOSES

The purpose of this research is to demonstrate the complementarity of intuition and logical thinking in a process of understanding mathematics basing on two basic notions of mental model and reflective thinking. In more concrete terms, we try to examine and identify students' mental models of a abstract and mathematical concept in regard to intuition, and observe how students think reflectively on their mental models in a whole-class discussion in regard to logical thinking. To attain the purpose, in this paper, we try to examine the validity of the two-axes process model, especially the horizontal axis of the model by analyzing an elementary school mathematics class in Japan.

THEORETICAL FRAMEWORK: THE TWO-AXES PROCESS MODEL

First of all, we must see the essence and characteristics of the two-axes process model of understanding mathematics. This model has been built as a result of the theoretical exploration in order to make the followings clear; Through what levels should students' understanding progress? How do students develop their thinking at each level of understanding? Naturally, the model consists of two axes, i.e. the vertical axis implying levels of understanding and the horizontal axis implying stages at each level.

In this model, on the horizontal axis, there is three learning stages, i.e. intuitive, reflective, and analytic stage. Those stages are originated in the work of Wittmann (1981) which emphasizes that three types of activity are necessary to develop a balance of intuitive, reflective, and formal thinking and that mathematics teaching should be modeled according to the processes of doing mathematics (p. 395). Koyama (1993) have modified Wittmann's definition of three activities in order to form a horizontal axis of the two-axes process model. Those three stages are described as follows (Koyama, 1993, pp. 70-71).

Intuitive Stage; Students are provided opportunities for manipulating concrete objects, or operating on mathematical concepts and relations acquired in a previous level. At this stage, they do *intuitive thinking*.

Reflective Stage; Students are stimulated and encouraged to pay attention to their own manipulating or operating activities, to be aware of them and their consequences, and to represent them in terms of diagrams, figures or language. At this stage, they do *reflective thinking*.

Analytic Stage, Students elaborate their representations to be mathematical ones using mathematical terms, verify the consequences by means of other examples or cases, or analyze the relations among consequences in order to integrate them as a whole. At this stage, they do *analytical thinking*.

Through those three stages, not necessarily linear, students' understanding could progress from a certain level to a next higher level in the teaching and learning of mathematics. As prominent characteristics of the two-axes process model, firstly, it might be noted that the model reflects upon the complementarity of intuition and logical thinking, and that the role of reflective thinking in understanding mathematics is explicitly set in the model. Secondarily, the model could be an useful and effective one which has both *descriptive* and *prescriptive* function in the teaching and learning of mathematics. The descriptive function means that a model can describe the real aspects or processes of the growth of students' understanding mathematics. The other is the prescriptive function of a model which can suggest us, researchers or teachers of mathematics, didactical principles regarding to the followings; what kinds of didactical situation are necessary, how we should set them up, and to which direction we should guide students in order to help them develop their understanding of mathematics (Koyama, 1992a, p. 181).

Those prominent characteristics of the model are, however, still expected theoretically. Therefore, we must examine both *validity* and ***effectiveness*** of the model in light of practices of the teaching and learning of mathematics. As a first attempt of such examination, in this paper, we will try to examine the validity of the model, especially the horizontal axis consists of three learning stages by analyzing an elementary school mathematics class.

A SKETCH OF ELEMENTARY SCHOOL MATHEMATICS CLASSES

The class to be analyzed in this paper is a part of four successive mathematics classes in a fifth grade (11 years old) classroom at the national elementary school attached to Hiroshima University in Japan. In February 1993, an elementary mathematics teacher of the classroom, Mr. Mori, planned and taught 36 students (18 boys and 18 girls) a topic named "Let's think with mathematical expressions". The students involved in those four classes are heterogeneous in the same way as a typical classroom organization in Japanese elementary schools, but their average mathematical ability is higher than that of other students in the local and public elementary schools.

In this section, firstly we see the intention of the topic held by the classroom teacher when he had planned it. Then a rough sketch is shown for an outline of four successive classes which actually developed in the classroom.

The classroom teacher, Mr. Mori, has a vision of elementary school mathematics education. Mori (1994) states it as follows: "Students' learning by solving mathematical problems is a continuous process of solving their own problems. I believe such process is an ideal form of learning elementary school mathematics that the once solution of a problem produces a more expansive problem (p. 91)". He planned the topic named "Let's think with mathematical expressions" with this vision of mathematics education. The main objective of the topic is to help students appreciate thinking with mathematical expressions such as interpreting a mathematical expression expansively and insightfully.

To realize this teaching objective, he planned three sessions and four unit-hour (45 minutes) classes for the topic as follows.

First session;	comparing lengths of two different semicircular roads	(2 unit-hour classes)
Second session;	comparing lengths of other geometrical figured roads	(1 unit-hour class)
Third session;	comparing areas of two different semicircular regions and summarizing the topic	(1 unit-hour class)

The followings is a rough sketch of an outline of four successive classes which actually developed in his classroom. In this sketch, students' activities are focused and picked up mainly.

First Class

1) Teacher set up the situation: "There are two places A and B. Let's make various roads between them". Students imagined and proposed their roads. Among them, semicircular roads were adopted and two different semicircular roads were drawn on a blackboard (Figure 1). One road L was a semicircular road with the diameter AB. Another road M was a one made by two connected semicircular roads with the diameter AC and BC, where place C was located at a

certain point on the segment AB.

2) Students predicted which road is shorter when comparing lengths of two roads L and M. At this point students had their own problem to be solved.

3) Students individually worked out the problem in their own ways. It must be noted that they had learned mathematical formulae for the length and area of a circle, and they know that circle ratio is about 3.14.

4) Students knew that two lengths of roads L and M are equal. Some students explained their own reasons of why two lengths are equal in the whole-class discussion. Students compared and interpreted those mathematical expressions written on a blackboard for the explanations.

5) Students compared lengths of two roads when place C had changed to be another point C' on the segment AB (Figure 2).

6) Students said their findings which they had been aware of in this class and proposed their own problems to be worked on in the next class.

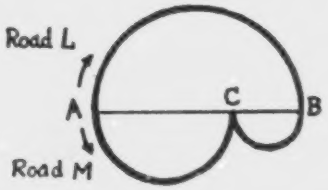

Figure 1.

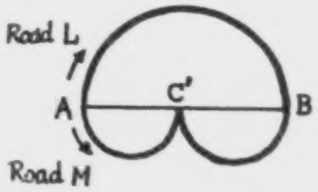

Figure 2.

Second Class

1) Students remembered what they had done in the first class.

2) Among the problems proposed at the end of the first class, students decided to work out the problem: "Compare lengths of two roads L and M when road M is changed to the one made by more than two small semicircular roads".

3) Students individually worked out the problem of comparing lengths when road M was made by three small semicircular roads (Figure 3).

4) Students presented their own solutions and compared mathematical expressions written on a blackboard in the whole-class discussion.

5) Students worked out the more general problem of comparing lengths when the number of semicircular roads of M increased (Figure 4).

6) Students said their findings which they had been aware of in this class and proposed their own problems to be worked on in the next class.

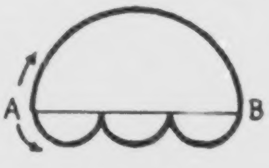

Figure 3.

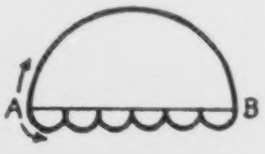

Figure 4.

Third Class

1) Students remembered what they had done in the second class.

2) Among the problems proposed at the end of the second class, students decided to work out the problem: "Seek for other geometrical figured roads which have a same rule as two semicircular

roads"

3) Students individually investigated two quarter-circular roads (Figure 5).

4) Students sought for other geometrical figured roads which have the same rule by means of mathematical expressions. Students checked, for example, two equilateral triangle roads (Figure 6) and two square roads (Figure 7).

5) Students said their findings which they had been aware of in this class and proposed their own problems to be worked on in the next class.

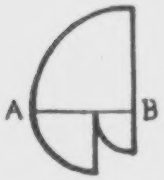

Figure 5.

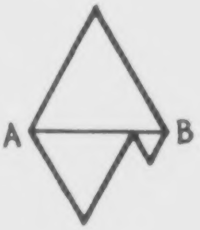

Figure 6.

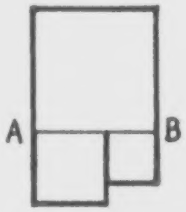

Figure 7.

Fourth Class

1) Among the problems proposed at the end of the third class, students decided to work out the problem. "Compare areas of regions encircled by two semicircular roads (Figure 8)".

2) Students individually worked out the problem with their own predictions.

3) Some students explained their solutions of the problem.

4) Students thought about how the area of region encircled by the road M changes when a point C moves from A to B on the segment AB.

5) Students represented the change of area in a graph.

6) Students read and interpreted the graph and explained their own findings about the change of area in the whole-class discussion.

7) Students looked back what they had done in all four classes and summarized the content of the topic named "Let's think with mathematical expressions".

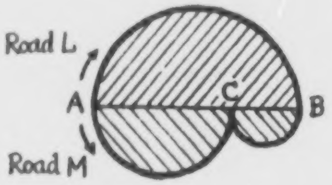

Figure 8.

DISCUSSION BY THE PROTOCOL ANALYSIS OF A CLASS

Four successive classes of the topic actually developed as shown in the above sketch. In this section, by analyzing the protocol of a class mainly in the first session, firstly we try to examine and identify students' mental models of length which lead to a misjudgement or a mathematically incorrect anticipatory intuition. Then we observe how their initial intuition has been changed under the control of students' reflective thinking in the whole-class discussion. Being Based on this analysis of a class, we examine the validity of the horizontal axis, i.e. three learning stages of the two-axes process model of understanding mathematics.

Identification of Students' Mental Models of Length

In the first class, after teacher's setting up a learning situation and students' discussion about mathematical problems to be solved, the process of teaching and learning actually developed as follows. In the following protocol of a class, sign *T* and sign *Sn* mean a teacher's utterance and a *n*th student's utterance respectively.

T: Today, we will try to work out the problem of comparing lengths of two semicircular roads L and M (Figure 1). How do you predict which is shorter, road L or road M?

S11: The length of road M is longer than that of road L, because the road M is bent at a point C.

S12: The road M encircles a smaller area than the road L does, so the length of road M is shorter than that of road L.

S13: The length of road M is shorter than that of road L, because the road M is closer to the straight line AB.

Those three students' utterances of their prediction allow us to identify their mental models of length which they have initially at the class as products of their previous experiences of learning length. S11 has a mental model like that when the both ends of two lines are trued up, a curved line is longer than a straight line as shown in Figure 9. S13 has a similar mental model to that of S11 like that because the shortest line between two points is a straight line, a line closer to the straight line is shorter as shown in Figure 10. On the other hand, noticing area, S12 has a different kind of mental model like that the length of a closed geometrical figure is proportional to the area of it as shown in Figure 11.

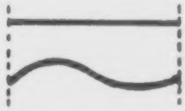

Figure 9.

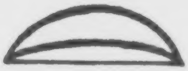

Figure 10.

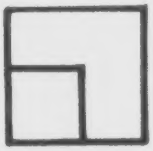

Figure 11.

All those mental models can lead to a mathematically correct judgement or prediction in some cases represented in figures 9, 10, and 11. However, in case of comparing lengths of two semicircular roads worked on in their class, their mental models produced a mathematically incorrect prediction. It might be said that they can not explicitly analyze the curvature (S11), closeness (S13), and similarity (S12). In any case, we could conclude that their mental models of length which they constructed previously and had initially at the class have a negative effect on their anticipatory intuition (Koyama, 1991) without any explicit analysis of their mental models.

Examination of the Validity of Three Learning Stages

Next, we will observe how their initial intuition has been changed under the control of their reflective thinking in the whole-class discussion. After students' predicting lengths, the process of teaching and learning actually developed as follows.

T: You have different predictions and your own reasons. Which is longer, road L or road M? Let's make it clear. Work out the problem in your own way and write it down on notebooks.

S14: I can not do, because we have no information about the length of AB.

T: Do you need to know the actual length?

SS: (Many students say "Yes", but some students say "No".)

T: If you need to know it, use that AB is 10cm and AC is 6cm.

SS: (Students individually work out the problem by using the mathematical formula for a length of circle which they know.)

T: OK! Present your own work to your classmates. Anyone?

S15: I calculated the lengths as follows. Two answers are equal.

Road L; $10 \times 3.14 \div 2 = 15.7$ *Road M;* $6 \times 3.14 \div 2 = 9.42$

$4 \times 3.14 \div 2 = 6.28$

$9.42 + 6.28 = 15.7$

S16: I can calculate the length of road M with one mathematical expression like this.

Road M; $6 \times 3.14 \div 2 + 4 \times 3.14 \div 2 = 15.7$

S17: I can do it more easily by using parentheses like this. Two answers are equal.

Road M; $(6+4) \times 3.14 \div 2 = 15.7$

S20: We do not need to calculate the lengths. The sum of AC and CB is equal to AB (looking at Figure 1), and we can see it apparently that both mathematical expressions for road L and road M is $10 \times 3.14 \div 2$*. So we can say that the lengths of two roads are equal.*

T: You have explained your works with your own reasons well. All of you seem to understand your classmates' explanations and be convinced them.

S21: Wait, Mr.! I have another idea. I used alphabetic letters. I thought about the problem when let the length of AB, AC, and BC be a, c, and b respectively. Then we can easily see that lengths of two roads are equal because two mathematical expressions are same like this.

Road L; $a \times 3.14 \div 2$ *Road M;* $b \times 3.14 \div 2 + c \times 3.14 \div 2$

$= (b+c) \times 3.14 \div 2$

$= a \times 3.14 \div 2$

In this whole-class discussion, with the explanation of S15 as a turning-point, students in this classroom reflect on their own calculating and thinking process and represent it in their own terms using mathematical expressions. This examination of the protocol allows us to conjecture that students do reflective thinking in their own ways. At this point, we should pay attention to the fact: S20 and S21 are explicitly aware that the mathematical expressions for lengths of two roads are same, while S15, S16, and S17 put their eyes on only that two answers are equal. In other words, for S15, S16, and S17 a mathematical expression is mere a thinking method to calculate an answer for comparing lengths, but for S20 and S21 the mathematical expression itself is a thinking object. This difference must be significant from a view point of the level of understanding mathematics, because, as van Hiele (1958) suggests us, the objectification could push students' understanding of mathematics up to a mathematically higher level.

In fact, the explanation of S21 stimulates other students and directs their understanding of this problem to a higher level, i.e. an understanding of the essential and mathematical structure of this problem.

T: It is a great idea. S21 used alphabetic letters. What can you see about the mathematical expressions explained by S21? Anyone?

S22: It does not depend on the actual lengths of AC and BC.

S23: They are expressed using alphabetic letters, so the lengths of two roads are equal even when a point C moves on the segment AB.

T: Is it true when a point C is close to the point A?

S24: Yes! As far as a point C is on the segment AB, two lengths are always equal.

T: Is it true? Please explain your reason in more detail.

(The following discussions are omitted.)

We can see in the above protocol that students do think about both the meaning of alphabetic letters and the structure of mathematical expressions. In other words, students in the classroom try to represent consequences of their reflective thinking more mathematically, analyze explicitly the structure of the problem, and integrate their findings as a whole. Therefore we might say that at this point of the class students do their analytic thinking.

CONCLUSIONS AND FINAL REMARKS

As a result of this observation and protocol analysis of the class, we see that the process of teaching and learning mathematics in this classroom actually developed in the line with the horizontal axis, i.e. three learning stages of the intuitive, reflective, and analytic which are set up in the two-axes process model of understanding mathematics. Therefore, we could conclude that the validity of three stages at a certain level of understanding mathematics has been demonstrated by the analysis of an elementary school mathematics class.

By the end of the first class, students in this classroom have become to be able to control their mathematically incorrect anticipatory intuition which they had initially at the first class by the logical thinking with mathematical expressions. It is saliently demonstrated by the fact that at the beginning of the second class 34 out of 36 students could predict correctly even when the road M is changed to be made by more than two small semicircular roads. This fact allows us to insist that as a result of their learning experiences students have a fairly determined intuition supported by the logical thinking with mathematical expressions including alphabetic letters.

In this paper, we have examined the validity of the horizontal axis consisted of three learning stages by analyzing an elementary school mathematics class. In doing it, we regarded students in a classroom as a whole and observed their process of understanding mathematics. It is, however, needless to say that we must also pay attention to an individual student and his/her process of understanding mathematics. Moreover, we have to examine the *effectiveness* of the two-axes process model of understanding mathematics in a sense that we can really make a teaching plan with this model and help students develop their understanding of mathematics to be an expected and higher level. Those are difficult but important tasks to be faced and addressed in our future research.

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Application of Reification Theory in Translating Verbal Expressions and Statements into Algebraic Expressions.

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This study presents a teaching model, grounded in Sfard's (1987) reification theory, for translating verbal expressions to algebraic expressions for students at seventh grade level and compares results with translation done in the more traditional, structural fashion. Six seventh graders participated in this study. The data collected suggested that the students initially perceived many concepts operationally. The table-filling method, building on their operational understanding of algebraic expressions, proved advantageous when translating verbal problems, especially for the average students.

Literature shows that historically algebra developed slowly over a period of 4000 years, from a time when all solution processes were done verbally and were mostly calculation processes until the sixteenth century when letters were employed as parameters allowing for a structural perception of algebraic expressions.

Reification Theory (Sfard, 1991) sees many similarities between the development of the perception of mathematical concepts and the historical development of algebra. This theory suggests that generally, a new mathematical concept is first grasped operationally (as a computational process only) and that the transition to a structural conception is a process that requires time and a cognitive effort. At an operational level "$2X + 4 + 3X$" might be perceived as an instruction to multiply a given number by 2, to add to it 4 and then to add to this sum three times the given number. A possible structural perception of this expression might interpret it as a mathematical "object", such as an unknown number or linear function. Many studies (e.g.Crowley, Thomas & Tall, 1994) have attested to the difficulty of structural perception of mathematical concepts. Eventually, the student should acquire the ability to perceive mathematical concepts on two levels - operational or structural, depending on the context The structural conception is the more advanced, and thus is more difficult to construct.

The Israeli student is first introduced to algebra usually in the second semester of seventh grade. The algebraic expressions are usually interpreted as generalized numerical computations which "illustrate a procedural [operational] perspective in algebra" (Kieran, 1992, p.392). Within a relatively short time the student is expected to think structurally in several areas of algebra. One such area is in translation of verbal statements. Many researchers (Chaiklin 1989; Clement, Lochhead & Monk, 1981; Lochhead & Mestre, 1988; Mestre & Gerace, 1986; Reed, Dempster & Ettinger, 1985) have studied the difficulties encountered when translating from the written language to the language of mathematics. Chaiklin (1989) sums up that most of the cognitive studies of algebraic problem solving testify to the great difficulty encountered by the student when encoding the relationships between the different magnitudes. Heretofore the student solved word problems by calculation processes, operationally, with no knowledge of algebra. Now he was expected to think structurally, to translate verbal expressions directly into verbal ones, to translate the think structurally, to translate verbal expressions directly into verbal ones, to translate the relationships between the different magnitudes which appeared in the word problem into equations or inequalities.

The Study

The purpose of this study was to examine a certain "operational" way of learning to translate verbal expressions and word problems and to compare results of translation done this way with results of translations as it is usually done, in a more traditional structural fashion. The assumption, grounded in Sfard & Linchevski (1994), was that the children start with operational conceptions, and the aim was to teach them translation, building extensively on their operational abilities, circumventing the need for well-developed structural-thinking abilities.

Six seventh grade students, boys and girls who learned in two similar, public middle schools, participated in this study. There were two average, two good and two very good mathematics students, evaluated as such by their mathematics teachers according to class and test performance. All of the students had just been introduced to algebra. They had been taught the concepts *variable, number coefficient* and substitution of numbers in simple algebraic expressions, as well as intuitive solving of equations. They had not learned any manipulation of equations, nor had they seen equations with unknowns on both sides.

Data collection

Each student initially answered a questionnaire in a combined oral-written interview in order to assess whether he perceived algebraic expressions structurally or procedurally. Examples of such questions are given in Table 1.

Table 1

1) You have to tell someone what is meant by **$5+n$** What would you tell him?

2) Before you is **$y-508 = 817$** What does this tell you? What can you do with it? Why?

3) What does **$4m+2m$** mean to you? What can you do with it?

These interviews allowed a closer inspection of the student's thinking. Thereafter each student was tutored individually in translating verbal expressions and statements using procedural methods. The number of sessions ranged between five and ten depending on the speed with which the student was capable of learning. All lessons were semi-structured, were conducted by the same teacher, and the topics and test questions were uniform across all students. The collected data are related to all interviews and lessons which were audio- and video-taped and transcribed.

Results of the questionnaire

The data collected suggested that many algebraic concepts were seen operationally. For instance, when Rikki Y., an average student, was shown the expression $y-x^2$ and asked what it tells her:

Rikki Y.: I have y which is any number and I take away from it another number to the power of two

Teacher: And what is this ($y-x^2$)?

Rikki Y.: It is an exercise with two unknowns, and they told us that y is any number less x to the power of two. And this x is equal to something and this gives us results.

Similarly, when a good student, Rikki S. was presented with the equation $12x+5=13x+4$:

Teacher: Have you seen an equation like this already in class?

Rikki S.: No

Teacher: What is the difference between this equation and the other equations that you have seen?

Rikki S.: That in the other equations which I saw she (the teacher) gave me examples like you gave me in the previous example ($28=5+x$), 28 is equal to some exercise. Here you gave me two exercises.

Teacher: All right. Can you do anything with it?

Rikki S.: I can try to find if there is, what x is equal to.

Rikki S. proceeded to successfully solve for x through trial and error by substituting the same number for x on both sides of the equations and calculating the numerical value of each side. Even a very good student, Maytal, displayed evidence of operational thinking.

Teacher: Does $8+4$ mean a number to you?

Maytal: Twelve.

Teacher: Yes. It means 12 to you, all right. What about $b+3$?

Maytal: It cannot mean a number because I don't know what b is. It is an unknown.

These exchanges are representative of the students' mode of thought at the time. Algebraic expressions, were seen as "exercises" namely as prescriptions for computational procedures. These procedures could be executed arithmetically, with appropriate numerical instantations. The one exception was Rotem, a very good student, whose language through all the interviews consistently suggested that his structural development was well on the way. He perceived $m+3$ as representing a number, spontaneously collected like terms, and was later able to translate relationships between magnitudes into algebraic expressions and equations with apparent ease using the traditional structural way.

Traditional methods of translating verbal relationships of quantities into the relevant algebraic expressions require a structural mode of thinking when translating into static descriptions. A simple example "One number is greater than another by 56. Their sum is 201. Find the numbers" requires the student to write:

First number: x ***Second number: x+56***

In the above symbolization, the student is expected to perceive $x+56$ as a number, when he cannot yet see $3+5$ as a number! (Compare episode related in Sfard & Linchevski, 1994, p.103). As has been illustrated above, there was little evidence for structural conception, whereas evidence for the operational approach abounded. Most of the seventh graders are still in the procedural stage of thinking. With this in mind, a teaching model was designed.

The Teaching Model

The method of translating word problems to algebraic expressions and equations proposed in this study is operational in nature and makes use of numerical instantations to clarify the written relationships. A typical early example in the tutoring might be "Find a number that is greater by seven than a given number". The student is instructed to clarify for himself or herself the meaning of the text using a numerical substitution process, to coin the given number, say **w**, and then write the required expression. Theoretically the table might look like that presented in Table 2.

Table 2

Given Number	Required number
-2	-2+7
0	0+7
-4.5	-4.5+7
w	w+7

Table 3

Given Number	Required number
-2	-2+7=5
0	0+7=7
-4.5	-4.5+7=2.5
w	w+7=

It is interesting to note that, in practice, when substituting numbers, the students were never satisfied with writing the required number as a sum, say **2+7**, but invariably calculated the result e.g. 2+7=9. They continued doing it even after experience showed them that the substitution process was but a tool for finding the relevant general algebraic expression, which itself could not be calculated further. Moreover, both average students not only calculated the numerical sums but also insisted on attaching an "=" sign to the derived algebraic expressions. Their table might look like the one presented above in Table 3. This illustrates the feeling of suspension that these students feel, that the expression **w+7** is incomplete, the phenomenon already noted by many researchers (e.g. Robinson et al, 1994). Clearly the students feel that the expression w+7 is but a prescription for a procedure and if one wants to speak about the result one has to perform these calculations. In other words, the students have not acquired yet the sense of duality of algebraic expression, of its representing a process and a result of this process at the same time. While filling the table the students were building on their operational understanding of an algebraic expression; they learned that wgeneralizing correctly requires finding the process not the outcome.

The prevalent errors in translating are performing a left-to-right word-order match and labeling, well known thanks to the famous "students and professors" problem (Lochhead & Mestre, 1988). The students learned that verbal expressions might mislead them into writing incorrect relationships and were taught how to check their translations. The students were encouraged to examine whether the 'required number' instantations that they had calculated indeed fulfilled the constraints directed in the translation problem; if the numerical outcome fit the bill, there was a very good probability that the algebraic expression would too. Obviously, if a student was still having problems translating number-wise, translating to algebraic expressions would prove difficult too. Initially, Tomer - an average student - fell into this category. The following is a typical exchange at this stage:

Teacher: What is a number greater by four than twelve?

Tomer: Eight. (He subtracted 4 from 12)

Teacher: And a number that is five less than minus two?

Tomer: Three. (He subtracted two from five)

Many times when faced with a familiar problem the students were tempted to translate directly, relinquishing the table-filling method. However, when faced with a problem that they were unsure of, they reverted to using number instantations. A case in point is in the translation of the "Student - Professor" problem. Five out of the six students answered this problem correctly, with no apparent difficulty. The two very good students translated directly, one good student and one average student used the table-filling method and translated correctly. Tomer, the other average student, at this stage was not able to find neither correct number instantations, nor the correct algebraic expression for this problem. The other good student showed a second's hesitation before producing his answer. The number of teachers is M, the number of students, S. This was the following exchange:

Amir: So in a school there are six times as many students. First of all it is multiplication and the number of teachers, so it is S times, no, M times six is equal to... (and he writes $M \bullet 6 = S$)

Teacher: Explain why you hesitated and how did you -

Amir: M are the teachers, this will be times six, so it is the teachers times six. Let's say you have five teachers, so five times six is equal to thirty. So this S expresses the students and there are thirty students.

Clearly Amir is convinced of the accuracy of his equation; the numerical instantations give him this confidence. The strength of this method may be clearly seen when we compare Amir's present confident performance with his lack of success in answering another test. This test adapted from Sfard (1987), was comprised of four translation tasks - two with structural answers and two with operational ones. Sfard (1987) reports that the success rate of the procedural questions was significantly higher than the structural ones. The following is the questionnaire presented to the students, as presented in Table 4.

Table 4

For every one of the following problems circle the correct answer

The Problem	Answers
1)In a certain class the number of girls is greater by 3 than the number of boys	a = the number of girls b = the number of boys (1)$a+3 = b$ (2)$a = b+3$ (3)$a - b+3$
2)The number y is 5 less than the number x	(1)$5+y = x$ (2)$y = x+5$ (3)$y < x+5$
3)In a certain class the number of boys is greater 4 times than the number of girls	To find the number of girls, one must (1)multiply the number of boys by 4 (2)divide the number of boys by 4 (3)it cannot be worked out
4)The number g is smaller, times 4 5, than f	To find g, one must (1)divide f by 4 5 (2)multiply f by 4 5 (3)it cannot be worked out

The English translations of the questions are somewhat stilted, due to our wish to preserve the right-to-left (Hebrew reading) word-order. The answers to 1 and 2 are structural in form; 3 and 4 are operational.

The students were informed that one of the three possibilities was correct, and they had to decide which answer suited the problem best. Since the questionnaire was multiple-choice, initially none of the students used the table-filling method, but rather chose the answers that seemed correct to them. The two very good, and one good student made all correct choices. After Amir, the other good student, had made his choices, most of them erroneous, the teacher asked him to do it again with numbers to be convinced which answers were correct. Below are excerpts of the conversations that accompanied the first and second attempts at answering the questionnaire. The first attempt is coined Trial I, the second - Trial II.

Trial I - Question 1:

Amir: (After reading the question) Because there are two (possibilities). $b+3$ is also equal to the number of girls. No, it has to be (1) (and encircles (1)). So here (pointing to (3)) it should be equal (not <). If it is equal is it right?

Teacher: Don't know, you decide.

Amir stays with his choice- (1) - and proceeds to the next question

Trial II (after the teacher suggested that he may use the table-filling method) - Question 1:

Amir: Yes, so there are ten girls and the boys are thirteen, aah, the other way around, thirteen and ten. (Reads his answer of Trial I): *a* plus *3* is equal to *b*. (Reads choice (2): *b* plus *3* is equal to *a*. What did I do?? I mixed the two up.

And he encircled (2) - the correct answer. Amir answered question 2 correctly in Trial I and confirmed it number-wise in Trial II.

Trial I - Question 3

Amir: (After having read to himself the question and all possible answers) One *can* work it out but not according to this. Is there something else (another answer)?

Teacher: What's given are these three (possibilities) and it seems that one of these is correct. If not, you can write "none of the above".

Amir: (after reading the question once more out aloud)...But the number of boys is not written and the number of girls is not written, (decisively) it can't be worked out.

And he encircles (3).

Trial II - Question 3

Amir: The boys are ten and the girls they don't tell me.

Teacher: (Reads the question) The number of boys is four times the number of girls.

Amir: So if let's say the boys are forty then the number of girls is ten, so divide the number of boys by four. Number (2).

And he encircled the correct answer

Similarly Amir was not able to arrive at the correct answer for question 4, and after deliberating with himself whether it could be (2) or (3), both incorrect, he chose (2).

In Trial II, question 4, even though he had worked operationally in questions (1), (2) and (3), he began to reason structurally:

Amir: Yes, I'll check it, g is four and a half times smaller... (Reading his answer to question 4) All right, it is all right.

Teacher: How did you check it?

Amir: Because times smaller is division, so it is the inverse operation, then multiplication. And here (he points to question 2) we do the inverse operation, then this is subtraction, this is plus (referring to his choice of answer), and here we did division, this (multiplying) is the inverse operation.

Teacher: All right.

Amir: Is it good?

Teacher: You checked it? How? With inverse operations, right?

Amir: Yes.

Teacher: All right. You know what, I'll ask you to check.

Amir: This number, let's say, g is smaller times four (reads to himself), how much is it? May I use it (the calculator)?

Teacher: Of course.

Amir: Four and a half times four is eighteen. This is four (writes "4" above g) and this is eighteen (writes "18" above f, reads his original answer) multiply f (pause). This (pointing to f) has to be greater that this (points to g) it is clear, this is eighteen and this is four. (Reads his answer again) multiply f by four and a half, wrong!

Teacher: So which is right?

Amir: Divide f by four and a half.

Teacher: Are you sure? You keep on asking me, now I'm asking you. Are *you* sure?

Amir: Divide f , it's eighteen, by four and a half, eighteen divided by four and a half is four, okay, yes

Teacher: Okay?

Amir: This time, sure.

As for the two average students, their first attempt gave only erroneous choices. Their second attempt, via table-filling, resulted in the three correct answers; both students found correct number instantations for Question 4, but chose an incorrect choice for this question on the questionnaire.

Discussion

This last episode illustrates again the strength of number instantations in translating, where abstract structural ways of reasoning may still cause the novice translator to stumble. An attempt to arrive at a general formula directly might result in inaccurate translation, whereas the systematic use of the procedural method could ensure correct translation. Trial 1, question 1, illustrates the well-reported error due to (in Hebrew) right-to-left word-order matching performed during translating, an error which is immediately corrected when the student uses the table-filling method. The number of instantations which each student needed was individual, depending on the complexity of the problem and his mathematical competence. A short while after being introduced to the table-filling method the better students relinquished it, automatically adopting the more traditional structural translating method, even though at no time was any suggestion made in this direction. However, when a difficult problem was presented, or if they erred, they reverted to number instantations and corrected themselves.

This study presents an alternative method of teaching cognitively immature novices to translate and provides the novices with tools for checking their results. The study demonstrated that using our operational method, greatly improved translational abilities when compared with the traditional translating methods, especially for the average students.

Further research and study is required to examine to what extent this method is applicable in more complex translation problems.

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MEASURES OF TEACHERS' ATTITUDES TOWARDS MATHEMATICAL MODELLING

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Abstract.

Mathematical modelling has recently become a compulsory element of the pre-university mathematics curriculum for approximately 70,000 16–19 year olds in the UK. Its implementation is highly uneven, and there is a need to measure its effectiveness in various ways, especially in the promotion of teachers' and students' attitudes. This paper reports on the development and validation of a scale: the mathematical modelling questionnaire (MMQ), to measure teachers' beliefs about the importance of modelling in the mathematics curriculum. Five factors were identified on the MMQ. Three of these reflect the teachers' definitions of the concept of modelling, relating to the reality of applications, the processes of modelling and communication. The other two factors relate to the importance of assessment and of technology.

Background

Mathematical modelling, here conceived of as the process of application of mathematics to 'real' problems outside mathematics, (see Niss, 1989) is now a growing part of the school mathematics curriculum in many courses all over the world (Watson, 1989; Burkhardt, et al 1990; Blum and Niss, 1991). But the depth of penetration and the method of introducing modelling courses into curricula vary. The UK, with a long tradition of incorporating real life problem solving activities and projects in school mathematics, adopted modelling projects piecemeal into some 16-19 courses over the period 1988-93. Two mathematics courses, the School Mathematics Project (SMP) and Mathematics in Education and Industry (MEI) which are at the forefront of this development with a student share of about 30% of 16-19 year olds, have shown how modelling can be integrated into such courses and assessed. Claims are being made that this has improved the motivation of teachers and students and hence the latter's take-up of mathematics as an option. A critical feature of the mass implementation of the new methods has been the development of assessment criteria and teacher training to support it (See Kitchen, 1993a; Kitchen and Williams, 1993).

Partly as a result, the UK has now adopted mathematical modelling in the compulsory core for all pre university mathematics programmes (SCAA, 1993). This now obliges all courses to adopt modelling, and to introduce it into their assessment schemes in some form. The approach to dissemination of modelling in UK mathematics programmes has thus now become a top–down one (Watson, 1989) and many questions remain unanswered. How will teachers respond to the new initiative? Do teachers believe mathematical modelling is important? Do they understand the concept and will they possess the necessary knowledge for its effective implementation? Will the greater use of modelling tasks emphasising group work, discussion, project work and investigation, termed the modelling approach,(see Kyeleve and Williams 1995) help to improve learners' attitudes ?

In light of these concerns, Blum (1991, p. 27) warned that:

> *All conceptions and proposals for mathematics teaching stand or fall with the teachers, with their professional abilities, didactical and pedagogical qualifications.*

Teachers' beliefs are known to influence their practices (Pajares, 1992; Ernest, 1989) especially in the implementation of new programmes such as mathematical modelling (Knapp and Peterson, 1995) which in turn influences their students (Peterson, et al 1989). Much has been said about modelling but little or no evidence exists concerning teachers' belief about the importance they attach to this component and how these are reflected in their classroom practices. Most studies of attitudes to maths problem solving have focused on standard problems, word problems or puzzles (Askew and Wiliam, 1995). Mathematical modelling is not about routine types of problem solving exercises, rather it is about real life problems in which mathematical knowledge and skills could be applied through modelling processes (Swetz, 1989; 1991).

As part of our effort to study the changing attitudes of teachers towards modelling in the curriculum, how they are influenced by the course they teach, their training and their context, it became necessary to develop scales to measure the strength of their beliefs about the importance of modelling. We expect that this instrument, and the methodology used to validate it, will be of interest to researchers and curriculum developers all over the world who want to evaluate system-wide change in this aspect of the curriculum, in which teachers beliefs will play a vital role.

Research design

A theoretical review identified a) the aims and pedagogy of teaching and learning the concept, skills and processes of mathematical modelling, b) summative evaluation criteria in modelling, and c) the inappropriateness of existing attitudes' scales for assessing teachers' attitudes to mathematical modelling. This led to the writing of the 37 items which were face validated by the authors and colleagues, after 9 were dropped this left the 28 items in the instrument (see appendix).

The key questions formulated were: 1). What are the major factors defining the teachers' perception of the importance of mathematical modelling and applications in the pre university mathematics curriculum? 2). To what extent will the teachers' understanding of the concept of modelling explain the factors identified ?

Subsequent steps adopted in the statistical validation of the instrument follow the methods of Rummel (1970) and Davison (1983). 1. The MMQ was developed, face validated and administered to a pilot set of 96 teachers using 6 pre University mathematics programmes in schools and colleges within Greater Manchester. 2. Interviews with 8 teachers who responded to the MMQ and observation of classroom teaching sessions were carried out to seek clarification on their understanding of the emerging factor structure. 3. The factors were interpreted by a group of mathematical modelling 'experts'. 4. The teachers' definitions of the concept of mathematical modelling were analysed using the emergent factors by one of the authors and by an educational linguist. 5. A second data set of 37 teachers from three contrasting pre university mathematics programmes drawn from a purposeful sample of 15 schools validate the pilot results. (The sampled schools were selected to minimise school effect in our further analysis of the influence of programmes, in-service courses and other variables on teachers' beliefs and practices as well as on the students' attitudes. The results of this study are in preparation.)

The MMQ is a five point Guttman type scale with 28 stems each with two items, (one asking for 'importance of' and the other asking for 'strength of classroom practice of') giving 56 items. All items of the questionnaires, identified from literature sources, were defined and framed to reflect the major characteristics of the modelling approach (Niss, 1989; Burkhardt, 1989; Blum and Niss, 1991; Savage and Williams, 1990; Haines and Izard, 1993; Kitchen, 1993b; etc). In section A of the MMQ with 28 items, teachers were asked to indicate their responses by ticking one of the five point options (unimportant to very important) coded 1 to 5 which represented how important they thought the item to be. In section B with a parallel set of items, the teachers reported how often they thought they practised the items on a 4 point option (never to often) coded 1 to 4.

Results.

The teachers'(N=96) responses to the items of section A of the MMQ which measures their beliefs were factor analysed using the techniques of principal component followed by orthogonal rotation. The analysis yielded 5 interpretable factor groupings, with 9 items dropped due to low loadings, experts' suggestions and reliability analysis (see table 1). Confirmatory factor techniques were employed with the second data set (N=37) based on the 28 items of the 5 factor groupings. The loadings of the items on the factors vary from 0.46 to 0.85 and these were consistent with the confirmatory analysis. The proportion of variance explained by each factor is given in percentages with factor 1 accounting for largest , i.e. 31% (see table 1).

The measure of the level of contribution of each item to the factor it belongs to (item-factor correlation) ranges from .48 to .89. The internal consistency reliability coefficients (Cronbach alpha) for all the factors are sound (.77 to .84) with that for the MMQ overall being 0.91. The reliability analysis based on the second data set is consistent with the earlier results for all the factors except for factor 2 which is lower. We further tested the 5 factor structure using the chi-square goodness-of-fit test which was found to be reasonable (chi-square = 340, df = 248, P< .000). Though small, the second data set also supported this 5 factor structure (chi-square = 290.3, df = 248, P< .033). In contrast to the approach of interviewing a sample of subjects first and using the findings to build these scales, we adopted the opposite approach since most teachers may not have developed adequate knowledge and appropriate attitudes about mathematical modelling (Blum, 1993). Nevertheless, we asked teachers to describe what they thought modelling is or ought to be (section C of the MMQ) and the results are also presented later.

Table 1 The MMQ Factor definitions, Standard deviations and Cronbach (alpha) reliability coefficients (pilot: n=96; main: N=37).

Factor (% of variance explained)	No. of	S. D		Cronbach (a)	
The Importance of:	items.	pilot.	main.	pilot.	main
F1: Assessing understanding & skills of mathematical modelling (31%).	8	3.84	3.92	.84	.85
F2: Real life Applications of Mathematics (9%).	6	3.75	3.00	.82	.65
F3: Communication skills in Mathematics(8%).	5	3.16	3.26	.77	.79
F4: Teaching and Learning Mathematical Modelling processes (6%)	5	3.23	3.03	.78	.79
F5: The use of Technology (6%)	4	3.24	3.00	.84	.87

Although the stems of factors 1 and 4 are essentially the same, the factors are different due to the presentation of the items. For all the items of factor 1 except one, teachers were specifically asked to: 'indicate how important it is to assess students on being able to'---, whereas for all the other items, teachers were asked to indicate: 'how important they feel about each statement' (see appendix). Consequently, factor 1 represents "the importance of assessment of understanding and skills of mathematical modelling in the curriculum" and factor 4 represents "the importance of mathematical modelling processes in the curriculum". Lester (1989) highlighted this problem of differentiating the teaching and learning of problem solving from its assessment: in our study this is measurable through the differences in the two factors.

Experts' opinion and definitions, and interviews with teachers

Of the 13 'experts' consulted, 8 produced titles for each group of items. In factor one, the terms 'modelling skills' or 'processes' appeared in five experts' definitions and three others defined it as 'problem solving skills'. The words 'real' or 'real world' appeared together with the word 'apply' or 'applications' in six of the experts' definitions for factor two. In factor three definitions, the key words 'communication skills' occurred in seven out of eight definitions. In factor four, the terms 'development, learning or teaching of' appeared with 'modelling processes or skills' in seven definitions given by experts, and factor five was unanimously termed 'the use of technology'. Therefore, the teachers' perceived importance of mathematical modelling in the curriculum, as measured on the MMQ, is defined by 5 factors which are called "Assessing understanding and skills of mathematical modelling", etc. as in table 1 above. The items belonging to each factor group are as identified in the appendix

Interviews with 8 teachers from the pilot sample indicated 'communications skills' as the major outcome of 'group work and practical activities' which they saw as part of the modelling curricular. Seven of the eight teachers interviewed felt that it is important to use group work and practical activities in the teaching of mathematical modelling which they perceived as being about encouraging 'oral and written communication skills in mathematics'. While teachers using courses (e.g. SMP 16-19) assessing skills of modelling processes expressed satisfaction with such courses, those using the more traditional maths courses were sceptical of the new initiatives, especially the assessment aspect. All the teachers expressed the need for training in the use of technology in modelling activities.

Fifty teachers gave interpretable written responses to what they thought modelling is or ought to be, such as:

T1. *Modelling is/should be the process where students (1) set up a model of a situation and consider the variables/limitations involved; (2) test a given law and gain evidence to support it rather than just accept it. (reflect F2 and F4)*

T2 *Modelling is translating a real life situation into mathematical concepts and quantities with the aid of various assumptions in order to predict or explain the outcome. (reflecting F2, F3 and F4)*

T3 *A mathematical model is a system in mathematics (perhaps expressed as a computer program) which mirrors a practical situation. An experimental model is a simplified version of the real life problem. (reflecting F2, F3, F4 and F5).*

The teachers' descriptions were categorised as either reflecting each of the descriptors for the five factors or not. For example, F2 was inferred from the use of words such as: 'real problem' or 'situation' (see eg T2), 'real life problems or situation', 'everyday problem' or 'physical problem'. The agreement between one of the authors and a non-mathematician educational linguist as to the assignment of definition statements to factors varied between 82% and 90%.

Assessment and use of technology were each reflected by 3 definitions. 11 definitions reflected communication skills. 36 and 38 definitions reflected real life applications and modelling processes respectively. Obviously, the 'assessment of understanding and skills of modelling' and 'use of technology' are not conceptual elements of modelling per se but important curricular factors to which teachers may attach importance.

Discussion.

Several approaches were used in identifying the factors of the teachers' attitudes to mathematical modelling as measured on the MMQ. These were based on (a) a priori estimate and fit; (b) interpretation from experts, interviews with teachers and analysis of their definitions; and (c) reproducibility of the factor structure based on the 'pilot' and 'main' data .

The factors of teachers' perception of the importance of mathematical modelling, as measured on the MMQ, do reveal a degree of common understanding about the importance of mathematical modelling with the community of 'expert', curriculum developers and trainers who have influenced the new curricula and encouraged the new core statement from SCAA in the UK. The reflection of the three elements in the teachers definitive statements (real application, processes of modelling and communication) about modelling in their attitudinal factors is interesting. It suggests there may be differences of understanding of the concept causing differences in attitude towards modelling, or they may be three aspects which are regarded as of greater or lesser importance by teachers. We note that some 'experts' may regard communication aspects as quite separate from modelling itself and note that only 11 out of 50 teachers gave communication skills as definitive. Indeed, unlike engineers, pure mathematicians may consider 'modelling' to be about translations within mathematics, and regard the 'reality' as irrelevant to their concept.

Of particular interest is the first factor of the MMQ which accounted for the largest proportion of the variation in the teachers' beliefs. Like technology, teachers see this as a separate issue. Assessment has always been a major factor determining the extent to which teachers may practice modelling in particular and any innovation in general. It is also natural that teachers see their practice of assessment as being detached from their personal beleifs.

The instrument has now been used to study the effect of training and courses taught on teachers' beliefs, establishing some links between the syllabus and the belief in the importance of assessment in mathematical modelling, for instance, and the lack of a link between attitudes and previous in-service training (See Kyeleve, 1994). The same methodology of factor analysis has further been employed in the reliability analysis of a student's attitude to mathematical modelling scale; it appears the factor structure is slightly more complex, but linked to the same factors identified in the teachers instrument.

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APPENDIX: SECTION A OF MMQ

Please indicate how important you personally feel each statement to be.	Unimportant	Fairly Unimportant	Reasonably Important	Important	Very Important	Factor Group
Maths should foster students' applied problem solving abilities.						2
Maths should enable students to become critical towards its use both in class and the real world.						2
Learning maths should prepare students to apply it in other subjects in the curriculum.						2
Maths tasks should represent a balanced picture of life in the world not just in the U.K.						2
Maths should assist students to acquire knowledge and understanding of mathematical concepts.						1
Students should develop the skills of oral/written reporting through individual presentions to their peers.						3
Students should be able to organise and participate in groupwork in mathematics.						3
Students should develop skills in the use of graphic calculators.						5
Students should develop skills in investigating and modelling mathematical concepts with computers .						5
Use of selected tasks should relate mathematics needed in daily life and in intelligent citizenship.						2
Maths teaching materials should establish links between maths and the real world.						2
Learning materials should help learners to pose problems and questions for their own investigations.						3
Mathematics teaching materials should illustrate the process of constructing new models.						4

Please indicate how important you personally feel about each statement.	Unimportant	Fairly Unimportant	Reasonably Important	Important	Very Important	Factor Group
The use of technical words such as: investigate, set up a model, interprete and validate should be taught and exemplified.						4
Teachers should use problem solving processes to introduce new mathematical concepts.						4
Teachers should illustrate the skills of planning, control, monitoring and evaluation in mathematics teaching.						4
Learners should acquire knowledge of modelling by setting up models, solving, interpreting and validating the solution.						4
Learners should experience solving maths problems using graphic calculators.						5
Learners should experience investigating mathematics concepts/models using computers.						5
Skills of listening, speaking, reading and writing should be integrated in all aspects of mathematics learning.						3
Assessing students in mathematical modelling should include practical and oral work.						3
Please, indicate how important is it to assess students on being able to:						
Understand a problem and express it as objectives to be achieved.						1
Consider those features which are relevant to the problem.						1
Justify and explain the assumptions made in the solution to a problem.						1
Select and define relevant variables with appropriate symbols and units.						1
Give a clear statement of the mathematical problem and find its solution to a desired accuracy.						1
Explain the solution within the context of the original problem, and revise the solution process if necessary.						1
Communicate and give a clear explanation of the problem and its solution process to the teacher or peers.						1

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INNOVATION-IN-PRACTICE: TEACHER STRATEGIES AND BELIEFS CONSTRUCTED WITH COMPUTER-BASED EXPLORATORY CLASSROOM MATHEMATICS.

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Abstract: *This is a report on research[1] into the strategies and beliefs constructed by eight teachers after six years of innovative practice supported by teacher education and involving a one-hour-per-week computer-based maths classroom activity by small cooperating groups of pupils. All teachers were observed for 3 teaching periods, verbatim transcriptions were made from video-recordings and semi-structured inteviews were subsequently taken. Combined qualitative and quantitative analysis indicates that the teachers had constructed idiosyncratic reflexive pedagogies which as a whole could be characterised by means of the type of pupil activity they intended to encourage, i.e. self-motivated interplay between reflective and directed activity with emphasis on the former regarding references to mathematics.*

Theoretical Framework

Research on mathematics teaching seems to have progressed from perceiving the teacher as the implementor of pre-prescribed teaching methods or innovations and interpreting teacher performance with respect to the extent to which the implementation has met its objectives (Hoyles, 1992). Interrelations between teacher pedagogy and teacher attitudes has come into focus, the latter disaggregated into attitudes towards the teacher's role, mathematics, the teaching of mathematics. The role of teaching process came into play, with distinctions like the one between teachers' espoused beliefs and those enacted in the classroom (Ernest, 1989) providing useful insights into the formative role of the classroom situation. The perception of a teacher constructing and reorganising a personal pedagogy through interrelation with classroom culture and the wider culture is emerging (Moreira and Noss, 1993), posing new methodological and theoretical tasks in providing interpretative frameworks. Olson's (1989) distinction between bureaucratic and reflexive curriculum change is based on teachers making sense of their environment as they act upon it. Lerman (1992) has called for the need to map relations between beliefs and beliefs-in-practice, termed "situated beliefs" by Hoyles (1992).

This is a report on research into eight teacher's strategies and beliefs constructed after six years of innovative practice supported by teacher education and involving a one-hour-per-week computer-based exploratory maths classroom activity by small cooperating groups of pupils. Our theoretical orientation in interpreting our classroom observations of teacher activity was that of the teachers constructing and reorganising pedagogy as they act upon it during their practice (Olson, 1989). We see knowledge as constructed through social interaction within cultures and feel that although we are best informed by combining Piagetian and Vygotskian theory, there is a lack of a theoretical framework focused on and deriving from the kind of teaching and learning interactions we observe (Mercer and Fisher, 1992).

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Background to the study

The study took place in the context of a longitudinal school project (Psychico College) involving a computer-based development of cooperative small-group investigation activity supported by teacher education (Kynigos 1992, Kynigos and Preen, 1995). From year 3 to 6 inclusive, all 500 pupils and 25 teachers take part. Each group carries out four-hour long investigations and prepares a written report to orally present to the rest of the class. Elsewhere, we have given brief descriptions of how centrally mediated information transmitting, exam-cramming processes characterise an encyclopedic, theoretical and content based Greek educational system (McLean, 1990). Not surprisingly (Noss, 1992) the advent of computers has enhanced rather than diluted these characteristics (Polidorides and Kynigos, 1993, Kynigos, 1995). Like Psychico College, some schools offer hours over and above those set by the system, in order to cultivate some creative and constructive activity in their pupils. It was thus in this framework that the "investigations" hour took place.

Teacher education was carried out by the researchers, was built within the teachers' working schedule and had as a main strategy to set up opportunities for the teachers to reflect on and discuss their practice and encourage developing pedagogies. The computer was used a) as a medium for expression and for generating exploratory activity and, b) as a "window" (Weir, 1986) to children's thinking for teachers and researchers and to teachers' strategies and beliefs as described in this report. More details on the background and on other research within the project can be found in Kynigos and Preen, 1995, Kynigos et. al, 1993. Studies of related issues in school settings can be found in, Hoyles and Sutherland, 1989.

Method

In the above setting, we investigated a) the teachers beliefs as constructed during this practice, regarding learning mathematics, their pedagogical role and that of the computer and b) their intervention strategies regarding the extent to which they were embedded in the pupils' investigations, the aspects of the learning situation they referred to, and the kind of pupil activity they intended to encourage.

Eight teachers were chosen so that their classes spanned all the age groups and were video-recorded during three teaching hours each. A remote microphone enabled transcription of all their utterances capturing the responses of the group of pupils in which they intervened. The person taking the recording was aware of the issues which could be interesting during the analysis. Semi - structured interviews were subsequently carried out regarding the teachers' views on the ways in which children learn during the "investigation" hour, how they perceive their own role and pedagogical strategy and how they compare this kind of pedagogy and learning to the one which goes on during the normal curriculum activities. Verbatim transcriptions of audiorecordings were made. Background data was also collected, i.e. all the pupils' written presentations of their investigations and researcher notes on specific aspects of each particular hour which may have influenced the atmosphere (e.g. a broken down computer).

Results

Comment Characterisations

We analysed the teachers' discourse into "notional units", giving each a characterisation according to our interpretation of a) whether it was embedded in pupil activity, b) to which aspect of the learning situation it refered to and c) the kind of pupil activity it intended to encourage (Kynigos and Preen, 1985 and for the latter, see also Hoyles and Sutherland, 1989). We avoided attempting to "objectify" the notion of "discourse unit", negotiating between ourselves to relate it to the characterisations themselves using pilot analyses of the same data to check for interpretative discrepancies. Fig. 1 provides a representation of this analysis showing how each embedded comment (A) was further given three characterisations (B, C, and D).

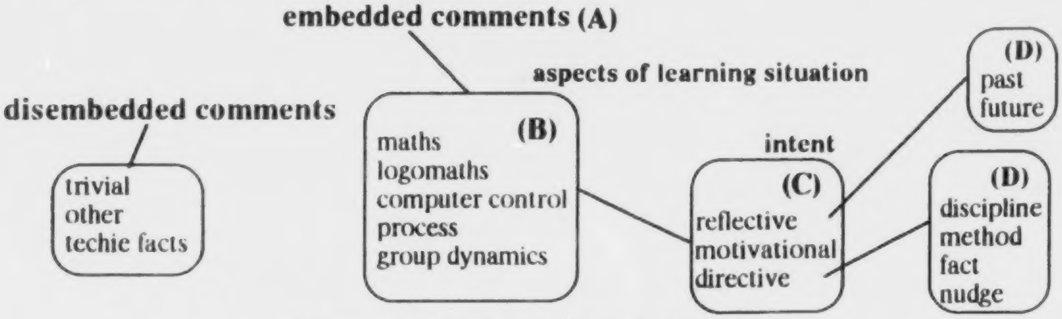
embedded comments (A)
aspects of learning situation
disembedded comments
trivial
other
techie facts
(B)
maths
logomaths
computer control
process
group dynamics
intent
(C)
reflective
motivational
directive
(D)
past
future
(D)
discipline
method
fact
nudge

Figure 1: Comment characterisations

So, for example, in the following pupil - teacher verbal exchange, we interpreted the teacher's comment as consisting of the following four units (marked u1, u2, etc): u1 = process, reflective, future, u2 = process, directive, fact (it would have been method if the comment was e.g. "haven't I told you to play turtle - in such cases as this one"), u3 = process, reflective, future, u4 = Logomaths, reflective, future.

> *pupil: "And how will I make it go like this?" (make the turtle turn from the perpendicular)*
>
> *teacher: "What do you say?" (u1) "Didn't I tell you to think you are the turtle?" (u2) "What would you do?" (u3) "Towards where would you turn to face that way?" (u4)*

We suggest that the information derived from the combined analysis of reference to aspects of the learning situation (as these aspects emerged from the data) and intended encouragement of types of pupil activity is helpful in describing teaching strategies as they are constructed during teaching practice. The aim was to gain insight into the ways in which the classroom culture and the dynamics of the situations emerging within each group of pupils interacted with teacher beliefs and teaching strategies regarding the above and mathematical ideas. Although these results are useful, there are limitations in their interpretative power. A major problem is the extent to which each comment can be connected to the context of the situation it was made in. It is not easy for instance to draw information on whether a comment refering to one aspect of the learning situation has influence on another aspect. The same applies for the types of activity. For example, taken out of context, the interpreter may characterise the following comment as refering to Logomaths, since the discussion seems to be about a turtle turn. Following all the interventions regarding these two pupils, however, revealed that the teacher was really trying to establish communication between them - so the comment refers to group dynamics.

> *"Wait a minute, wait a minute. Andoni, whenever you think of something do you just do it or do you communicate with the others? Because just now Nikiforos was puzzled, (he asked) "It 15?", as if he did not know what you were going to do."*

In characterising the comments we thus took into account these contextual issues to the extent that the video recordings made possible. Finally, this analysis is considered in conjunction with, on the one hand, vignettes taken from one or a series of episodes, and on the other, the building of a quantitative picture of each teacher and all the teachers as a whole.

A vignette

Teacher A has had 12 years working experience at primary level. She has taken part in the "Investigations" project from the start, and had taught third and fourth grade students during the six years of the project's duration. She did not have a mathematics qualification more than that provided by her primary teacher's degree. She herself does not feel confident with what she terms "mathematics" as an object of study. However, she believes that there is another, natural, everyday kind of maths from which school teaching diverts pupils to perceiving it as alien territory.

> *"I believe that maths is in our life, in ourselves and we do it subconsciously - but someone comes and says to us: "look, what you were doing till now is fine, but I will teach you to do it differently like this and this and this", so you don't do it at all and you say: "ah! mathematics is difficult, that's it, I cannot do it"."*

Regarding the meanings she brought to the project, she did seem to make specific connections with her pedagogical aspirations to encourage cooperation and autonomy amongst pupils.

> *"I thought it was very important when I was given the chance to teach them to cooperate, to make some decisions on their own and to try to understand what they are doing and why."*

In describing her strategies, she saw herself as offering services to problematical situations already arisen. That is, when her pupils or herself have identified a problem hindering further activity, then comes the intervention.

> *If they call me I usually go, if they don't I just walk around and when I see that they are stuck.... I ask "what's going on, what's the problem?"*

She further feels the need to "explain" her directedness and the urge to provide pupils with answers, indicating internal conflict on the issue of controlling her interventions.

> *"Many times the answer comes out naturally, its difficult to hold yourself."*

In the following episode, teacher A intervened on her own accord after a group of two third-year pupils had taken some time reiterating fd 10 lt 15 fd 10 lt 5 and at some point changing to fd 10 lt 10 fd 10 lt 5, in order to construct a planet to go with their rocket project and after the teacher had initially encouraged them to try to make a circular planet not letting them settle for a square one.

> *T: "Have you come to some conclusion? (yes) What?"*
>
> *P: "To make these sides lt 15 and lt 5 and those here lt 10 and lt 5"*
>
> *T: "Ah, so not to have the same lt everywhere, ok, try it, but can you think beforehand and imagine more or less what shape will come out?"*
>
> *P: "It will not be exactly a circle. In some parts it will be rather straight"*
>
> *T: "Ah, then it's worth thinking about the turnings again, since here (points to screen) with these commands you don't get large straight bits, but she goes and turns bit by bit, while with these commands you get large straight bits... maybe you should consider the commands again and instead of you getting a long eggy shape with straight bits you can get something more round? Have a look, compare these bits which get you quite a round bit and these which do not get you much of a round bit. Don't delete old commands, it will confuse you, yes leave them so you can check. To see, for example, what happened there, where we changed the lt's what changed in the shape? Or where our lt's were the same, what was the shape like? "*

The teacher's agenda seemed to have been for the pupils to investigate how to make a circle and to progress to the "right" answer, which she was clearly aware of. Reading her comments gives the impression that she is internally struggling between providing too much information and steering the activity towards constant turns. Even though she will accept constant turns (and not necessarily the classic lt 1 fd 1) as a didactical goal and attempt to not dissociate her intervetion to the pupils agenda for making a planet e.g. by referring to "round" and "egg-like" shapes, she seems "pushed" by the situation - time constraints, poor pupil results, lengthy lapsed time of pupil investigative inertia. She seems to be impatient with the lack of an exploratory culture - she perceives that the pupils do not reflect, check commands against their results of the screen, compare sets of commands, enjoy hypothesising and making an effort to make a more circular shape. So she "tells" them to do so, in one instance. In fact, their agenda seems quite different from the one aspired, or expected by the teacher, and the fact that after this episode they simply ignored the comments, typed a few more commands and went on to write en essay on how great their rocket was (and not a word about the planet) is a clear enough indication. So, in attempting to encourage investigation, the teacher in effect gave a relatively large number of "technical directions" hoping with this one-off intervention to influence pupil activity from then on.

The quantitative picture

The total number of comments (around 3,500) allowed a quantitative picture to emerge regarding individual or collective information between comments, teachers and year-groups. Three types of analysis were carried out: a) the aggregate relative frequences of all categories for all the teachers as a whole and for all the observed teaching periods of each teacher individually, b) statistical tests for the significance of the distance between individual and aggregate values for teachers and comment characterisations and c) tests for significance in differences between specific categories of interest and amongst teachers for these categories. Some such results are presented at this point.

embedded comments		**intent**		**maths and logomaths**		
				reflective past	13	20
maths	5	reflective	40	reflective future	7	
logomaths	31	motivational	6	motivational	2	
computer control	19	directive	44	directive discipline	2	13
process	25			directive method	4	
group dynamics	10			directive fact	4	
				directive nudge	3	

Table 1: Aggregate relative frequences of comment characterisations (%)

The following observations are made: a) the embedded comments (90%) by far outweigh the disembedded ones, b) there are few motivational comments, implying that there was little need for teacher - prompted motivation, c) we do not have a significant difference between reflective and directive comments on the whole, but have significantly more of the former when it comes to comments with specific reference to maths and logomaths and d) regarding the reflective comments in the latter two categories we have significantly more of those referring to the past. In general, we see a large percentage of comments with no reference to maths (at least 54%), which at least supports justifying the characterisation with respect to the aspect of the learning situation and provides some indication of the nature of the discourse and classroom culture.

In order to investigate individual teachers' and comment characterisations' deviations from the aggregate picture, we calculated the *expected* values (the values corresponding to the aggregate score), did a chi-square test for the differences between observed and expected values and then tested for the significance of each difference individually using a special test taken from biometrics (Haberman, 1973). We then observed the significance of this difference in two ways: "horizontally", in order to study variations amongst comment categories and "vertically" to do the same with variations amongst teachers. The results show that we have large variations regarding the aspects of the learning situations and insignificant variations regarding intent. For example, in the former comment category, only 1/8 teachers were close to the aggregate for the procedure and group dynamics categories and 4/8 and 3/8 for the maths and logomaths categories respectively. Moreover, regarding these four categories, only two teachers were close to the aggregate in three of them, and three teachers varied significantly in all four. The rest were close only with respect to one aspect. With respect to intent, the picture changes, as shown in table 2, where on the left we have the ratios of teachers with insignificant differences to the aggregate scores and in the centre and right the ratio of intent categories for which each teacher did not vary significantly from the aggregate.

directive	5/8	teacher 3a	3/3	teacher 5a	2/3
reflective	5/8	teacher 3b	1/3	teacher 5b	2/3
motivational	6/8	teacher 4a	3/3	teacher 6a	2/3
		teacher 4b	3/3	teacher 6b	0/3

Table 2: Cases of insignificant variation from the aggregate scores

Furthermore, as pupil grade increases, the results show a decrease in maths, group dynamics and process, while there is no such trend regarding logomaths. Regarding maths this could indicate that the teachers relate less and less school maths to exploratory mathematical activity as the pupils' age and the school maths content changes. This may point to the need for further reflection on the part of the teachers on how to help the pupils synthesise understandings emerging from this activity to school content (Hoyles and Noss, 1993).

Conclusions

We suggest that characterising teacher comments by combining aspects of the learning situation they refer to and kind of pupil activity they intend to encourage is useful, especially with the informative role of their quantitative handling and the illuminating support of qualitative analysis and vignette construction. However, there is need for more focused methodologies for systematic ways of connecting results from varying methods such as the above. The results describe teaching innovation as acted out and constructed in the classroom by teachers supported by systematic but not intense or directive teacher education. Qualitative analysis supports the view that espoused beliefs can be different to enacted ones, but more importantly, that individuals' actions may be influenced by belief systems, but not necessarily by one, or in any systematic prescribed or reproducible way. Aspects of the situation, classroom culture and wider culture may have central bearing on activity. Analysis of the characterisations of teacher comments indicate that the teachers had constructed idiosyncratic reflexive pedagogies which as a whole could be characterised by means of the type of pupil activity they intended to encourage, i.e. self-motivated interplay between reflective and directed activity with emphasis on the former regarding references to mathematics. To end, we suggest that providing time slots which can play the role of outlets in the system for trying out alternative methodologies may help generate more reflexivity amongst teachers.

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THE IMPLEMENTATION OF CURRICULUM POLICY ON CLASSROOM ORGANISATION IN PRIMARY MATHEMATICS IN CYPRUS

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ABSTRACT

This paper presents findings, from a larger study of Cypriot primary teachers' perceptions of curriculum reform in Mathematics, concerning the ways used by them to organise their classroom. Questionnaires were distributed to randomly selected sample of 10% of all Cypriot teachers. No group of teachers organised their Mathematical lessons in order to distribute their time equally between working as a class, working on individual tasks and working on collaborative group tasks. Relationships between teachers' perceptions of curriculum reform in Mathematics and the ways use to organise their classroom were identified. Statistically significant differences in the ways use to organise classroom were associated with years of teaching experience, and characteristics of the class taught. Implications for the process of change in Cyprus are drawn.

I) Introduction

Curriculum reform has not proved easy to effect in many countries partly because its success is not dependent on the substantive content of the reform alone (Howson 1991, Rudduck 1991). The failure of much curriculum innovation has been attributed to the neglect by innovators of teachers' perceptions. Fullan (1991) has argued that the reasons for the failure of most educational reforms goes far beyond the identification of specific technical problems. He supports the argument of Wise (1977) that policy-makers are frequently "hyper-rational" and points out that:

> "innovators need to be open to the realities of others: sometimes because the ideas of others will lead to alterations for the better in the direction of change, and sometimes because the others' realities will expose the problems of implementation that must be addressed and at the very least will indicate where one should start" (p. 96).

On this analysis, the process of curriculum change in Mathematics should be conceptualised not only in terms of teachers' abilities to implement the reform but also by reference to their perceptions of teaching Mathematics.

The importance of the role of teachers' perceptions derives from examining the effectiveness of the models of curriculum change which have been developed by several commentators (Havelock 1971, Schon 1971) to explain the systematized change. In Cyprus, the process followed for the design and diffusion of curriculum change has been a "centre-periphery" model, operating in what is a highly centralised system. The failure of centre-periphery model has been attributed to the fact that teachers' perceptions were typically inadequately considered at two important stages; the adoption, ie the teachers' decision to use an innovation, and the implementation, ie its realisation.

This paper is concerned with some of the findings of a larger study into Cypriot teachers' perceptions of curriculum reform in Mathematics. It is an attempt to present the findings concerned with the implementation of curriculum policy at classroom level and especially with how Cypriot teachers organised their classroom. The reasons for dealing with how teachers organised their classroom rather than how they thought that a Mathematics classroom should be organised has to do with the policy requirements that the time for teaching Mathematics should be distributed equally in working as a whole class, individual tasks and collaborative tasks (Ministry of Education 1994). Cypriot policy documents referred to this as "balance".

II Methodology

In April 1993, questionnaires, designed to identify perceptions of reform policy on curriculum and assessment in primary Mathematics, were sent to randomly selected sample of 10% of Cypriot teachers. A response rate of 72% was obtained. Semi-structured interviews with 20 teachers who responded to the questionnaire were also conducted in order to test the validity of the questionnaire findings by matching the qualitative data derived from interview with each teacher against the quantitative data gathered by his/her individual questionnaire. A measure of match was derived by comparing most of the questionnaire with the interview data gathered by this study. Although this measure does not necessarily imply that its validity is high since it is possible that they are both invalid, the use of both questionnaire and interview methods provides a basis for triangulation of data. In using these two methods, I could also collect both quantitative and qualitative data to explain more fully and study from more than one standpoint teachers' perceptions of curriculum reform in Mathematics.

III REPORT AND ANALYSIS OF RESEARCH FINDINGS

This section is divided into two parts. The first one deals with the findings arising from questionnaires. A comparison between data derived from semi-structured interviews with 20 teachers and their own responses to the questionnaire is provided in the next section. Interview data are also used to assist in interpreting the questionnaire findings.

A) Questionnaire Findings

An item of the questionnaire asked teachers to estimate the proportions of time in Mathematics lessons that their pupils spent in working on individual tasks, on collaborative group tasks and as a whole class. Cluster Analysis was used to identify relatively homogeneous groups of Cypriot teachers according to their responses to this item. Means and standard deviations of the ways used by the whole group of Cypriot teachers and those of the six groups of teachers to organise their classroom derived from Cluster Analysis are shown in Table 1.

Table 1: Means and standard deviations of time in Mathematics lessons that the pupils of six cluster groups and of the Whole Group of Cypriot teachers spent in working on individual, on collaborative group tasks, and as a whole class.

Group of Teachers	Whole class		Collaborative tasks		Individual tasks	
(Number)	Mean	S.D.	Mean	S.D.	Mean	S.D.
Type I (N = 38)	83.45	6.30	9.37	5.55	7.13	3.91
Type II (N = 51)	48.00	4.01	31.29	6.11	20.00	6.07
Type III (N = 9)	21.00	6.00	22.78	6.67	56.11	7.82
Type IV (N = 42)	66.78	5.61	24.76	8.18	9.88	2.81
Type V (N = 34)	60.50	9.09	14.65	6.72	26.03	8.54
Type VI (N = 8)	21.00	4.43	58.75	9.54	20.00	9.26
Whole Group (182)	59.04	19.17	22.96	13.52	18.30	13.94

The following observations arise from Table 1. First, most teachers (91%) organised their Mathematics lessons in such a way that children spent more than 50% of their time in working as a whole class. This is revealed by the fact that pupils of teachers who are members of most of the clusters (4 out of 6) spent most of their time in working as a class. There are only nine teachers (Type III) who organised their class in such a way that pupils spent more than 50% of their time working on individual tasks and eight teachers (Type VI) who organised their class so that pupils spent more than 50% of their time in working on collaborative group tasks. Second, there is no group of teachers which organised their Mathematical lessons in order to distribute their time equally between working as a class, working on individual tasks and working on collaborative group tasks.

We can now describe each group of the six-cluster solution on the basis of their responses to this item. Teachers of the first group (N=38) are those who spent most of their time in working as a class (80%) and the rest of the time was equally distributed to individual and collaborative tasks. However their pupils were very rarely involved with either collaborative or individual tasks. Their practice was mainly based on a teacher-centred approach to teaching Mathematics. They can be called as the "whole class instructors".

Teachers of the fourth and fifth groups were also "whole class instructors". They organised their classes in such a way that their pupils were working as a whole class for more than 60% of their time. However, pupils of the fourth group had also the opportunity to work on collaborative tasks (25%) and pupils of the fifth group spent 30% of their time in working on individual tasks.

The second group of teachers organised their classes in such a way that their pupils spent less time in working as a whole class than pupils of the above three groups. Their pupils did not spend more than 50% of their time in working as a whole class. This is the group of Cypriot teachers who organised their classes relatively close to the suggestions of the policy documents emphasising the importance of distributing their time equally to the three ways of classroom organisation.

The third and the sixth groups are those which are consisted of the few Cypriot teachers who did not spend most of their time in working as a whole class. The third group of teachers (N=9) based their teaching in Mathematics on individual tasks (50-63%) and the rest of their pupils' time was equally distributed to collaborative tasks and whole class tasks. These teachers can be called as the "individualisers". On the other hand, teachers of the sixth group (N=8) organised their teaching Mathematics in such a way that pupils were mainly involved with collaborative tasks (50-67%) and the rest of their time was equally distributed to whole class and individual tasks. We can call them "group workers".

Relationships between teachers' perceptions of curriculum reform in Mathematics and the ways used to organise their classroom were explored. Correlations about general aims for the curriculum in Mathematics and methods of classroom organisation were identified. There was correlation between support for the general aim of Mathematics related to mathematical communication and the amount of time which pupils spend in working as a group ($r=.61$ $n=183$ $p<.01$). There was also a negative relationship between the perceived importance of this purpose and the amount of time spend in working on individual tasks ($r=-.53$ $n=183$ $p<.01$). Significant relationships between teachers' responses to items concerned with different issues of Mathematics pedagogy and their responses to the item concerned with classroom organisation were also explored. There was a correlation between agreement with the opinion that there is a fixed sequence of Mathematical topics and the amount of time which pupils spend in working alone ($r=.56$ $n=181$, $p<.01$).

Finally, I attempted to identify whether there is any association between teachers' characteristics measured by the independent variables of the questionnaire and their responses to the item concerned with how they organise their classroom. Statistically significant differences ($p<.05$) in the ways used to organise their classroom were associated with years of teaching experience, and characteristics of the class taught (ie class size, and pupils' year group) but they were not associated with differences in their professional training.

B) REPORT AND ANALYSIS OF INTERVIEW DATA

Teachers' comments in response to my open question on approaches to teaching Mathematics raised the following issues concerned the classroom organisation for teaching Mathematics. First, all of them mentioned that they spent most of their time in teaching the whole class, a finding on the questionnaire also.

There was also a match between their responses to the interview and a ranking of these three ways of classroom organisation which was done on the basis of their responses to the questionnaire item.

Second, there were twelve teachers who acknowledged that they spent most of their time in working with the whole class, but this was not because they considered it as the most appropriate approach. A gap between teachers' perceptions of teaching Mathematics and their curriculum practice was identified. Seven of them took the view that working in group tasks took pupils more time to cover Mathematical topics and to adjust to this way of working. The following comment echoes this perception:

> "I like to provide opportunities to my pupils to work in collaborative tasks but I can not do it so easily. I do not have much space in my class to arrange tables in groups. But what is the most important obstacle of working in groups is the content of the curriculum which we have to cover. This cannot be achieved, if we use this approach. If I were not under stress, I could use group work more often and work with the whole class less often" (Teacher B.2)

The other five teachers indicated that they wanted to spend more time on individual tasks, but that they did not have adequate time to teach all these topics of the curriculum following this approach. Manageability problems and problems with their planning were also seen as obstacles for any attempt to use the individual approach for teaching Mathematics and especially since they did not have many resources to individualise tasks. The national textbooks were seen as not helpful. Furthermore, they acknowledged that spending most of pupils' time in working as a whole class was appropriate only for pupils who are neither low nor high attainers. They, finally, argued that although the content of the curriculum could be taught to the whole class by spending most of pupils' time in working as a whole class, this might not help low attainers to learn it and high attainers to learn as much as they could. That is to say they saw it as leading to difficulties in differentiation.

In response to my last open question concerned with ways of improving curriculum practice, three teachers considered having smaller class size than now as the most important way of improving practice. They argued that they saw this as related to a focus of classroom organisation, in which pupils can spent most of their time in working on individual tasks.

V) DISCUSSION: IMPLICATIONS OF RESEARCH FINDINGS FOR CURRICULUM REFORM POLICY IN CYPRUS

The evidence presented above can be discussed in terms of its implications for the implementation of curriculum reform in Cyprus. The research data reveal that there is no group of Cypriot teachers who organised their classes close to the suggestions of the policy documents emphasising the importance of distributing their time equally to the three ways of classroom organisation. Working as a whole class is the dominant way of teaching Mathematics in Cyprus. Evidence about classroom practice in England revealed that although teachers are encouraged by the policy documents (DES 1989, 1991) to organise their classroom so that their pupils will spend their time equally on working in individual tasks, in group tasks and working as a whole class, English pupils spent most of their time in working alone on a Mathematics tasks (DES 1978, Galton et al 1980, Barker-Lunn 1984, DES 1992). It can be therefore claimed that although classroom practice in Cyprus is different to that found in England, teaching time is not equally distributed to these three ways of classroom organisation in either England or Cyprus. These findings raise doubts about whether curriculum practice can change merely by the publication of national curriculum documents which mechanistically encourage teachers to spend their time equally on individual, group and whole class methods of organisation.

However, only very few Cypriot teachers organised their classroom in such a way that pupils spent all of their time working as a whole class and none of the cluster groups revealed comprised teachers using only one approach. This is in line with studies on English curriculum practice which reveal that the exclusive use of a single teaching method was rarely found (eg DES 1978, Bennett, 1976; Barker-Lunn 1984, p. 179).

The qualitative data revealed also that none of the 20 teachers interviewed, agreed with spending most of teaching time in working with the whole class but their practices were dominated by this way of classroom organisation. This conflict between the teachers' ideal version of practice and their actual practice has implications for their occupational culture. Cypriot teachers in recognising the discrepancy between their beliefs about classroom organisation in Mathematics and practical realities attributed it to the pressure of time arising from an overloaded curriculum. Both experienced and beginning Cypriot teachers considered the content of the New Curriculum in Mathematics as difficult for their pupils to understand (Kyriakides 1994). It can be claimed that the implementation of policy on classroom organisation will not have to face barriers of ideological kind (Howson et al 1981) from teachers. The main source for the gap between perceptions of teachers and their practice may lie in the way the system is operating and particularly policy requirements in respect to the content of the curriculum they have to teach in their class. This led teachers to organise their classroom in such a way that pupils spent most of their time in working as a whole class even when neither the teachers nor the policy documents consider this as the most appropriate way of classroom organisation.

It can be also claimed that barriers to the implementation of policy on classroom organisation in primary Mathematics of Cyprus may lie in the high degree of central control at school level through national textbooks, a national curriculum specifying the content of the Mathematics curriculum to be taught to each age-group of pupils and a specified length of curriculum time. These contribute to a mismatch between the ideology promoted by the curriculum policy and the administration of the system. And although teachers' perceptions of classroom organisation were similar to the policy requirements, the fact that this control did not promote flexible classroom strategies limited the policy's effectiveness.

It is also important to examine the policy on classroom organisation in relation to the other issues raised by the policy on curriculum reform in Mathematics. Teachers considered working conditions such as class size and the lack of resources as significant barriers in implementing the policy on classroom organisation. However, there is no intention by the government for reducing the size of the classes. Thus, teachers may not be encouraged to give more thought to the best way to respond to individual learning needs. In addition, teachers believe that national textbooks are not designed in order to enable them to provide different mathematical tasks for pupils who have different learning needs. Nevertheless, there is no intention by the Ministry of Education to publish new textbooks or to provide teachers with relevant resources to enable them to provide each child with learning experiences which take account his/her characteristics and learning needs.

Policy on purposes of Mathematics seems to be an important aspect related to the policy on classroom organisation. Although a low priority was given by Cypriot teachers to the role of language in teaching Mathematics (Kyriakides 1994), the development of pupils' ability to communicate by using Mathematics is not considered very important purpose of primary Mathematics by the New Curriculum. However, a correlation between the perceived importance of this purpose and the amount of time pupils spend in working in co-operatively group tasks was found. Thus, barriers to the implementation of policy on classroom organisation in relation to the use of group work may lie in teachers' perceptions of purposes of teaching Mathematics. It is therefore important that Ministry of Education, in addition to concentrating on the importance of using Mathematics in communication, should also provide teachers with more specific ways to organise their classroom in working on group tasks and at the same time provide opportunities to children to communicate by using Mathematics.

It should be, finally, indicated that the consideration of balance in classroom organisation by policy documents seems to be problematic. It should not be seen in terms of Mathematical proportion. The critical notion of classroom organisation is that of fitness for purpose. Teachers need the skills and judgement to be able to select and apply whichever way of classroom organisation - class, group and individual - is appropriate to the task in hand. The judgement should be educational and organisational rather than a matter of mathematical proportion (Alexander et al 1992). A balance between class, individual and group work is necessary not only because of the need for a variety of approaches but also for the fact that each one fulfils different purposes.

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PARTITIONING AND UNITIZING

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The goal of this study was to better understand the development of children's unitizing processes. Using a cross-sectional design, the study analyzed the partitioning strategies of 346 children from grades four through eight in terms of a framework that translated economy in the marking and cutting of the shared objects into sophistication in unitizing. At each grade level, a greater percentage of students used economical partitioning strategies than used less economical cut-and-distribute strategies. As grade level increased, the percentage of students using economical strategies increased, indicating a shift away from the distribution of singleton units toward the use of more composite units.

Background

There is a consensus that as one encounters the domain of rational numbers, changes in the nature of the unit largely account for the cognitive complexity entailed in linking meaning, symbols, and operations (Hiebert & Behr, 1988; Behr, Harel, Post, & Lesh, 1992; Harel & Confrey, 1994). Unitizing is the cognitive assignment of a unit of measurement to a given quantity; it refers to the size chunk one constructs in terms of which to think about a given commodity. For example, given a case of cola, one could think of it as 24 cans or (1-unit)s, 2 (12-pack)s, or 4 (6-packs). The ability to form and operate with increasingly complex unit structures appears to be an important mechanism by which more sophisticated reasoning develops. Research in proportional reasoning, for example, indicates that one of the most salient differences between proportional reasoners and non-proportional reasoners is that the proportional reasoners are adept at building and using composite extensive units and that they make decisions about which unit to use when choices are available, choosing more composite units when they are more efficient than using singleton units (Lamon, 1993a, 1993b).

Research in other rational number domains (Pothier & Sawada, 1983, 1984; Hunting, 1983; Kieren, Nelson, & Smith, 1985) highlights partitioning activities as important mechanisms for building rational number understandings (Piaget, Inhelder, & Szeminska, 1960; Pothier, 1981; Kieren, 1976, 1980; Streefland, 1991). The ability to divide an object or a set of objects into equal parts appears critical to the logical development of part-part and part-whole relationships and notions of equality and inequality, and may influence children's understanding of other mathematical topics such as measurement and geometry (Pothier & Sawada, 1990).

Together, the bodies of research concerning unitizing and partitioning suggest that the two processes build different and essential perspectives toward the understanding of rational numbers. Partitioning is an operation that generates quantity; it is an experience-based, intuitive activity that anchors the process of constructing rational numbers to a child's informal knowledge about fair sharing. Unitizing is a cognitive process for conceptualizing the amount of a given commodity or share before, during and after the sharing process.

The purpose of the study reported in this article was to make children's tacit unitizing process explicit through partitioning activities, to define more clearly the relationship between the two processes, and to connect children's partitioning and unitizing strategies to given contexts. By taking snapshots of children's partitioning strategies at each grade level, fourth through eighth, it also sought to identify trends or stages that characterize the development of increasingly sophisticated unitizing ability.

The hypothesis underlying this research is that the mathematical power afforded by the notion of equivalence should, for concrete operational students, also work to help them to compose units. Growth in sophistication of the unitizing process, signified by the use of more composite units or larger units, should be reflected in students' partitioning processes. After the partitioning stages characterized by Pothier and Sawada (1983), additional stages should emerge in which partitioning strategies grow increasingly economical.

Procedure

Children from grades 4 through 8 were given 11 tasks in which they were asked to draw pictures to show how they would share various types of food among given numbers of people. Five children were chosen at random from each grade level to participate in standardized clinical interviews. They were given the same tasks in the same written format as the rest of their classmates, but were asked to think out loud while they were working so that the researcher could gain further insights into their methods and reasoning.

Sample

The 346 students who participated in the study came from three schools in two midwestern cities. Three intact classes from each grade level, grades four through eight ($n_4 = 63$, $n_5 = 60$, $n_6 = 72$, $n_7 = 69$, $n_8 = 82$) from one parochial (K-8) and two public schools (K-6 and 6-8) participated in the study. Each class was heterogeneously grouped and culturally diverse. All teachers reported that they had never given their students partitioning activities.

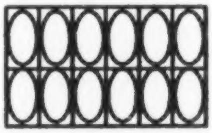
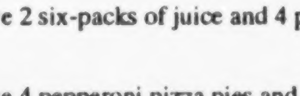
Tasks

The partitioning tasks used in this study are described in Table 1. The tasks were designed to include several known and hypothesized influences on children's partitioning activities. In addition to the usual distinction between discrete and continuous elements, tasks were differentiated according to the manner in which food items are packaged. For example, the packaging of food items may suggest or constrain certain partitionings, as in the case of eggs packaged in standard arrays in cartons, and sectioned candy bars. Composite units such as 6-packs of Coke or packs of gum may be left intact or opened to reveal individual items. Differences in partitioning were also expected when the items to be shared were alike, such as several pizzas all with the same topping, as opposed to sharing pizzas with different toppings or several different Chinese dinners.

Framework for Analysis

The Behr et al. (1992) semantic analysis represents idealized models of partitive divisions. Children's actual partitioning activity often involves other details. For example, to divide 3 pies among 4 people, some children cut each of the 3(1-unit)s into 8 parts and then distribute the pieces. Sometimes there are differences in the way children mark fractional parts on the objects and in the way they actually cut them. For example, when the whole consists of 4 (1-units) to be partitioned into 3 shares, each of the(1-unit)s may be marked as sixths and cut into thirds, or in three of the (1-unit)s, halves may be marked, and only the fourth (1-unit) might be marked and cut into thirds. This study is concerned with some of those finer details because they provide insight into the sophistication of children's unitizing process and dependence on perceptual supports. To account for these distinctions, partitioning is defined in this study as determination of

Table 1
Partitioning Tasks Used to Study Unitizing

Characteristics	Task
discrete subsets separable array form	You have the carton of 12 eggs pictured below, and 3 people who want to eat them for breakfast. 
discrete subsets separable composite form	You have 5 packs of gum and 4 people. (A pack of gum has 5 sticks of gum inside.) You have 8 six-packs of cola and 3 people. You have 2 six-packs of juice and 4 people.
continuous elements dissectible like items	You have 4 pepperoni pizza pies and 3 people. You have 4 chocolate chip cookies and 3 children. You have 4 oatmeal cookies and 6 children.
continuous unlike items people to eat dinner.	You have 1 cheese pizza, 1 mushroom pizza, 1 elements dissectible sausage pizza, and 1 pepperoni pizza for 3 people. You have 3 Chinese dinners (1 pork, 1 beef, and 1 chicken) and 6 You have 4 Chinese dinners (1 pork, 1 beef, 1chicken, and 1 seafood) and 3 people for dinner.
continuous subsets separable prepartitioned	You have the 2 candy bars shown below and 5 children 

equal shares and is viewed as a multi-stage operation: marking objects, cutting them, and clearly indicating one person's share.

Pilot studies were used to classify children's fair-sharing strategies. These studies revealed that strategies may be differentiated along at least four dimensions: (a) preservation of pieces that did not require cutting in cases where each person receives more than one object in a discrete quantity; (b) economy of the marking (not using sixths when thirds suffice); (c) economy of cutting (not making more cuts than necessary); and (d) the nature, packaging, and social practices related to the objects being shared. Based on the these characteristics, a hierarchy of 9 partitioning strategies (described more fully in Lamon, 1996) was used to code student work. In general, a higher level of sophistication in the unitizing process involves the ability to use more composite units, and in the context of these activities, is indicated by the preservation of pieces that do not require cutting and by economy in making marks and cuts. That is, decreasing sophistication in unitizing may be denoted by the increasing fragmentation of a share of the food. However, an exception occurs when customary practice supersedes concerns for economical marking and cutting, as would be the case when sharing three pizzas, each with different toppings, among three people. The obvious and most economical way to share is to give each person one pizza, but it is customary for each person to have some of each pizza.

Results

When partitioning wholes consisting of continuous, dissectible, like elements, such as pizzas of the same type, the data show trends toward economy from grade 5 to grade 8, and within grade, a preference for economical marking and cutting. When the composite wholes were composed of unlike elements, such as Chinese dinners, students cognitively differentiated the situation from one in which like items were shared. In this case, they favored a strategy in which all pieces of the whole were marked and cut and a piece of each was distributed to each person. Their strategies were heavily influenced by social practice related to the commodity being shared.

When partitioning composites of discrete items, such as packs of gum or six-packs of beverages, although the results still showed an increasing ability from fifth through eighth grade to use economical strategies, a greater number of students at each grade level had a stronger need to see individual items when using prepackaged discrete elements to be sure that each person was getting the same number of pieces. The students who required this perceptual support drew the packs of gum (a unit of units) with the five sticks showing. Students tended to use economical strategies , even in the case when they are cutting arrays and prepartitioned items, such a cartons of eggs or sectioned candy bars. Although these items might present constraints or tend to suggest certain cutting patterns and discourage others, most student strategies reflected an effort to partition using as few cuts as possible.

The results of this study characterize children's unitizing process as one that develops over time and with experience in varied contexts. Decomposition of a given whole into small units appears to happen immediately and naturally in the course of fair sharing activities, but reunitization into composite pieces, shown operationally as greater economy in marking and cutting fair shares, develops less rapidly. In this study, students engaged in intuitive activities in which they experienced amount, not merely number; the relevant question switched from "How many?" to "How much?" and their partitioning strategies illuminated some of the stages they went through as they applied and extended the number sense and counting processes that had served them through whole number concepts and operations in order to make sense of more complex operations that generate quantity.

This study also highlights the subtle and often tacit interaction of context and cognition as it affects the shape of one's mathematics. It showed that student partitioning strategies were situationally specific and showed a strong observance of social practice and practicality. The role of visual cues in mediating the step to unit composition was more salient in these written activities than it could have been in real life sharing activities. For example, students were able to see that they could group two one-sixth pieces of pizza and cut as if they had marked thirds. In the end, a more composite unit was used than the student may have intended at first due to a visually induced operational equivalence. Students were "experiencing" equivalent amounts.

This study suggests that partitioning has not been fully exploited as a didactic device for helping children to develop rational number ideas. For the teachers whose classes were involved in this research, either partitioning was a foreign notion, or it was considered a third grade introductory fraction activity. This study strongly invites partitioning activities into the middle school curriculum and, in order to develop sophistication and versatility in unitizing, encourages their sustained use until students have attained economical strategies across a wide variety of contexts.

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Simultaneously assessing intended, implemented and attained conceptions about the gradient

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Abstract

The research question guiding this study was: What are the conceptions about gradient effectively ***attained*** *in a specific calculus course, how are they* ***implemented*** *and how are they related to the conceptions that the curriculum* ***intends*** *to convey? A group of four graduate and two undergraduate students decided to investigate and modify their own mathematical conceptions formed and being formed in calculus and analysis courses. One teacher agreed to conduct tutorial meetings, whose reports were taken as research data. The conclusion was that the* attained *students' conceptions are related to the* intended *curriculum conceptions in a fragmented manner. The conceptions' fragmentation appear to be* implemented *by the demand imposed on the students by the use of a text book, whose analysis indicated that an obsession with loosing control over neglected infinitesimals, ends up hiding the essence of the physical and geometric properties of the gradient under the heavy mantle of mathematical rigor.*

Research question and theoretical references

In late 1995, a group of four graduate students in mathematics education showed dissatisfaction with their mathematics conceptions formed along undergraduate calculus and analysis courses. They decided to investigate and overhaul these conceptions. One teacher accepted to coordinate the group. Two undergraduate (sophomore) students joined in. The group realized the possibility of investigating simultaneously the *intended*, the *implemented* and the *attained* curriculum (Robitaille and Dirks, 1982). The subject of gradient was chosen, since it was the topic being addressed in the calculus course in which the undergraduate students were enrolled at the time. A research question as in the abstract of this article was agreed upon.

The meetings aimed at understanding what was known by the students about gradient, and how it was coming to be organized in the simultaneous calculus course. The meetings focused "on conditions under which students will choose to modify, reject, or extend their conceptions" (Confrey, 1990, p. 22). Strike, Hewson and Gertzog (1982, reference ibid. p. 22) "require that a student be dissatisfied with an existing conception and find a new conception intelligible, plausible and fruitful", in order that accommodation can occur. In the case of this research, dissatisfaction with existing conceptions was granted from the beginning and the other conditions were supplied along the way. According to Piaget (cited in Confrey 1995, p. 4) "knowing an object does not mean to copy it - it means acting on it". It can be inferred that, in order to know student's conceptions, it is necessary to act on them, producing changes through a teaching practice. Such was the guide-line of this research. For the final remarks we drew on psychoanalytical theory (Lacan).

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Research procedure

A weekly forum of discussion and reflection was started. In our initial planning meeting, it became clear that, whether the participants had seen *gradient* three or more years ago, or whether they had seen it that morning, their spontaneous conceptions[7], centered in the *belief-statement* "vector of partial derivatives" would not enable them to sustain a dialogue about such questions as: "What is it used for? What does it mean? What is the geometric representation? Give an example." In brief, there was no other *sense* for *gradient* than the strict mathematical *meaning* of the definition. They did not "remember" any physical or geometrical meaning associated with this word.. To understand how this "forgetting" occurs, it seemed important to investigate the context (classroom, exams, textbook, teacher, etc.) in which the meanings are negotiated during the functioning of the Calculus II course and the subsequent Mathematical Analysis courses. This was partially done and is reported bellow.

At the beginning of each session, the participants carried out the exercise of talking about what had occurred in the last session; that is to say, the entire process was reviewed in the form of a synthesis. The coordinator intervened to point out the mathematical lapses in the discourse of the person who was speaking, posing questions. At any given moment, someone would be unable to contain themselves and go up to the blackboard to justify what they were saying. When that did not occur, someone was designated by the group to fulfill this role. When someone grasped a point that was being made before the others did, that person was asked to guide the others through questioning in the same manner as the coordinator.

A diary was kept of the first sessions that was presented to the coordinator who made some editing comments. These comments were discussed by the group, with the aim of producing a final script in the form of an article. A fragment of this diary is reprinted below. The teacher's comments, in brackets, reproduce part of his personal notes, taken during and after the sessions. Throughout this process, there was an ongoing attempt at teaching guided by the mathematical concepts themselves; however, the teaching practice was always subordinate to the learning. At no time did the coordinator push beyond what those present were able to comprehend, as evidenced by the justifications they made, even when this tactic lead to long periods of silence. The following basic rule was always upheld: If you do not know, explain; if you do know, ask questions. Only on the occasions of synthesis did the person who knew explain. A psychoanalysis inspired principle was followed: it is through speaking that one learns and through listening that one teaches.

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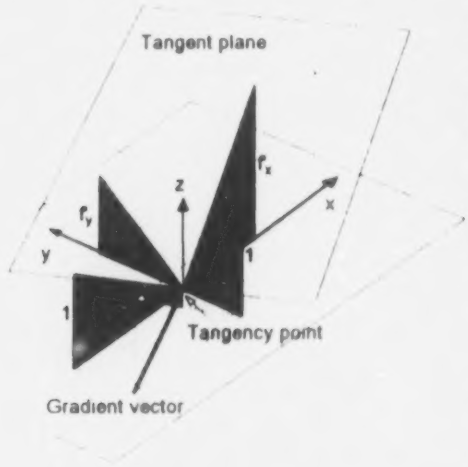
[7] "Spontaneous conception" refer to student's conceptions before the research started. For the uses of the term "conception" see Confrey (1990)

The attempts at justification outlined by the students led the group to draw the perspective of graphs of real functions with two variables, especially the design of the paraboloid, which functioned as a model for all the properties of the gradient. The difficulty in recognizing the points and curves in the perspective, always precariously drawn on the blackboard, led the coordinator to construct a model, initially improvised using the hard cover of a notebook but, the following week, constructed from acrylic boards, representing the coordinate planes, the tangent plane and lines, the normal and the gradient. From the moment this model was detached from the paraboloid, it became possible to enunciate and justify the properties of the gradient as beliefs of the group, applying only elementary geometric reasoning to the model: the gradient is the projection of a lower normal to the tangent plane on the xy plane, is a vector that is perpendicular to the level curves and has length equal to the largest growth rate of the function at the point. The question then came up: Where does the textbook (Guidorizzi, 1986[8]) state these properties, and how does it proceed to justify them? Why had they escaped the students in their careful page-by-page reading during the Calculus course? The participants then began by locating the statements about these properties in the textbook and examining the book in retrospect to know what steps the author took in an attempt to justify them.



Commented report of first meeting: the paraboloid (August 21st, 1995)[9]

The teacher asked if we had seen the gradient in Calculus II. We responded that we had. *What is it, the gradient?*, he asked. We wrote $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. *Give an example*, he requested. After some reluctance, we proposed the function $f(x,y) = x^2 + y^2$. *What*

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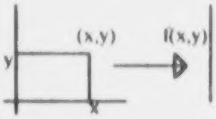
[8] In fact, this is a Brazilian author, but similarity of Calculus textbooks indicate that any other choice would probably have led to the same results.

[9] Taken from the diary of two undergraduate students.

does this function do?, he asked. The question surprised us. *What do you mean, what does it "do"?* we replied. *Well! Functions function, they have a domain and a counter-domain. How does that one work?* We responded that we had no doubts that it was a paraboloid. *Then draw it*, he said.

[They did not return to me the conception of function as a correspondence. A diagram like this one on the side does not appear to form part of these students' spontaneous conceptions about functions. It would not help to "explain" or "talk" about this since, throughout their schooling, these student must have heard these "speeches" about "correspondences" innumerable times. I prefer to continue and come back to this point using the graph of the paraboloid as a point of departure.]



The teacher suggested that we mark the point (3, 4, 25) and think about the graph of the function.

[They are not seeing the link between the drawing and the points they marked.]

The following definition of graph came up:

$$Gf = \left\{(x, y, z) \in \Re^3 | z = f(x, y); (x, y) \in \Re^2\right\}$$

We finally made explicit the set of level curves.

[It has been a lot of work for me to get them to read the definition as "the set of points (x,y,z) such that...". Remark added later: It is interesting to note that, in the report, there is a reference to something else: to the "set of level curves", not to level curves as sets of points. That would have demanded they use the concept of inverse image, and that would demand that they think of functions as correspondences.]

The teacher asked us to point out the values assumed by the function at the points of the curve of "level one". He emphasized: *the curve of level one*. At that moment we had doubts since, even though we had written the definition of graph, we were not using it. Upon analyzing c_1 we perceived that a function assumes constant values at all points of the curve $x^2 + y^2 = 1$. By the end of the session, the definition of level curve was clear to everyone.

[We did not go back to the gradient; the drawing was never corrected with respect to parallelism; the point (3,4,25) was never located as the intersection of the paraboloid with the planes x = 3 and y = 4.]

[The function $x^2 + y^2$, which should be an example for studying the gradient, became a problem in itself: There was no discussion of a correspondence, but of a "paraboloid". The design of the graph, which should lead to recognition of the domain and counter-domain, became another problem. The sketch of the ellipses inscribed in parallelograms, necessary for the perspective, would have led to the problem of graphic representation of tangents, which I preferred not to get into. The marking of points in the horse-back perspective, which should have aided the sketching of the graph, turned into a time consuming problem of careful measuring. Resorting to the level curves to fix the graph did not work. The level curves were yet another problem. In the meantime, the definition of the graph of a function was evoked in a surprisingly immediate fashion. It was remembered (but not used) and generated a new problem: What does the definition have to do with the "mug"? The memories associated with the equation of the circumference, as a function of two variables, did not lead them to put the level curve on the plane, much less to evoke a cylinder in space.]

[Throughout, the imagination had the model of the mug as a reference. The paraboloid, as a geometrical object, occupied the first plane of attention. "The paraboloid? Sure, we don't have any doubts about it. We know what it is. Axes? Oh, yeah. You have to design axes on the paraboloid. You have to mark the points on the paraboloid. You have to design the level curves of the paraboloid. Finally, you need to connect the definition of "graph of a function" with the paraboloid." One sees that, in the absence of the conception of function as correspondence, which should be the function as a **principle**, the spontaneous conceptions are generated by nucleation of **models**; in this case, the paraboloid.]

[The nature of the fragmentation of the spontaneous conceptions is now somewhat more clear. There is a nuclear model that the student grasps onto and to which the schemes that are stored in different "files" refer, ready to be used there, and only there. In the absence of this model, the "file manager" is lost, and he/she only "remembers having seen it".]

[It is conjectured that it is not the student who is fragmented, and that the problem is neither understood nor resolved with considerations of cognition only. The fragmentation is in response to a specific demand of the university which is fragmented into required courses, optional courses, and exams. In the next course, the paraboloid will no longer be considered; therefore, all that was hanging on it will be without a control center. The "file manager" will be lost. "Why learn another way of marking points on the graph if we already know how it looks like? Why learn to draw inscribed ellipses in a parallelogram if there are no exams in this course?" they seemed to say.]

Commented report of the analysis of the textbook (November 5-19, 1995)[10]

Through our thoughts and reasoning about the acrylic model, and using only the resources of elementary geometry, everything seemed very clear to us. We had summarized our understanding with the following points:

- the gradient is the vector of partial derivatives (definition)
- the gradient is perpendicular to the level curves
- the gradient points in the direction of the greatest growth of the function
- the length of the gradient is the rate of growth of the function in its direction

Not one of these properties seemed unusual to us. Now we were certain that we had already heard about them, that we had seen them in classes and earlier readings. We were curious to know how these properties that looked so familiar to us, were discussed and demonstrated in calculus books like the one used as a textbook in the course in which two of us were enrolled.

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[10] Taken from the report of two graduate students in Mathematics Education

We started by locating the perpendicularity statement and went searching backwards in order to find out how it was demonstrated and on what previous results the author made it to it depend. We looked for the essential nature of the proposition hoping for the transparency that we had obtained through our reasoning about the model. We found that the corresponding material in the textbook is complicated; it consists of a series of premises and statements that led us astray when we tried to trace back the theoretical path leading to the properties. We concluded that it was necessary to read the book from the first page. These fundamental statements appear "en passant" and, to justify them, the author lays out a virtual arsenal of theorems, including the chain rule, internal products, and even, unnecessarily, the implicit functions theorem! Finally we found a text of U. D'Ambrósio about the *mystification of knowledge* that best expressed what we felt. We can affirm that, in our study group, the phenomenon of demystification of knowledge occurred in a very clear form. It was actually a study of ethnomathematics.

"Rarely does anyone argue that the origin of knowledge resides in the people and obeys a very specific socio-cultural context. The explanations given for this knowledge are naturally partial and at times it appears with an apparent lack of coherence and comes impregnated with a strong *mysticism*. This knowledge generated by the people passes through a process of structuring and coding that, afterwards, is expropriated by powerful groups. In this way, this same knowledge (...) originating in the people becomes accessible to them only in a structured, coded form, most of the time subjected to *mystification* that results in institutional processes of devolution, such as schools, professions, academic grades, and the whole series of training mechanisms. The executors of the devolution to the people of these diverse bodies of knowledge should be recognized by the same power structure, in such a way as to secure their ideological commitment. This credentialing occurs by way of a system of filters; the individual normally loses sight of the process by which they are being coopted and which goes from the mystical, normally present in the origin of knowledge, to the mystified, as though this same knowledge presents itself to be dressed in a system of codes" (D'Ambrósio, 1989).

[The future teachers, as social agents, should offer guarantees, just as the textbook guarantees all of the steps presented therein. Nonetheless, it is necessary that they "lose sight of the process", precisely to better exercise their guaranty function. For this, the book was perfect.]

[The operation of [11] separating from the surface of the paraboloid, the plane and the tangent lines made concrete and tangible by the model, did not impede the students. From the moment we substituted the surface by a plane and the coordinate curves by tangent lines, Euclidean geometry took care of the rest and everything became clear. On the other hand, the author seems determined to keep under vigilance the infinitesimals that are to be overlooked, making explicit the errors that tend to zero when divided by their respective increments. It appears that in this book, as in most calculus books, a true obsession with loss guides the entire presentation and ends up hiding the essence of the physical and geometric properties of the gradient under the heavy mantle of a premature control of the mathematical rigor. In vain, the students try to *make sense* of what they read. They can only follow the book line by line and check the rigor of the mathematical *meaning*. The organization of didactic books is, after all, one of the factors that allows us to explain the fragmented appearance of spontaneous conceptions. The obsession with the control of the overlooked infinitesimal institutes an *accountability of loss* that takes precedence over learning.]

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[11] This is basically an operation of overlooking the second order infinitesmals

Final words: some psychoanalytical inspired remarks

Initially, some of the participants demonstrated a relationship with mathematics somewhere between *afflictive* and *fuzzy*. As they have already graduated from college and could be doing something else with their time, for one reason or another, they chose to be there and prolong or relive this old relationship; in this we should recognize a repetition that, for Lacan, is "the sister of *juissance*". It has to do, then, with a *symptom*. A *transferential* (affective?) *relation* installed itself at that point that made it possible for these participants to enter into the experience of learning. It was expected that the coordinator would occupy the position of speaker, the subject-supposed-to-know.

The desire of the participants was, certainly, sustained in fantasies which we did not intend to delve into, except for those which were undeniably present in the process. As to what lay beyond *selective listening*, the "*fantasy of omnipotence through the domination of knowledge*" (Walkerdine, 1988, Chap. 9) was present "*- Now it is so clear. Why did we not learn this in this way?*" exclaimed the students. The symptom brought by the image of the severe father, the one that awakens the hysterical, revealed itself during the process, when one of the participants verbalized "*- I keep looking for responses to give to him (the coordinator)! Actually I don't have to do that.*" She sat down but a bit later she was back at the blackboard. This same symptom revealed itself again at the end, in the report of a dream of another participant who went to a party at the moment a third one was struggling to resolve a problem that was necessary to write this article: "*- I dreamed I was at a party and he (the coordinator) was there looking very angry. He was sitting on a sofa with a computer in front of him, writing . .* ". At the beginning, clearly, everyone thought they should know the answers to the questions that were asked. Later they began to verbalize that not knowing was irrelevant. Finally the "I know/I don't know" problematic disappeared. Meanwhile, one question remains open. To the point that psychoanalysis projects itself beginning with selective listening, it remains to be seen if this could be the correspondent to what lies beyond fantasy?[12].

The affliction and the fuzziness go on. Although they show comprehension more quickly when asked about the subjects worked on at the beginning, fuzziness is still present when a new subject is addressed. It seems as though they incorporated affliction and fuzziness as their way of being with respect to mathematics. We can say that they identify themselves with affliction and fuzziness, whose memory they have so much fun with. They no longer appear worried about freeing themselves of these symptoms; on the contrary, we would say they enjoy them. Affliction and fuzziness became the "high" of the group. The student who dreamed about the "severe father" later revealed that *"- I didn't tell you everything."* She recently broke up with her boy-friend of two years, who told her *"- You're not the same person I started out loving."* There is a passage there, forbidden to the teacher, that only the psychoanalyst can transpose. This virus is truly dangerous.

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[12] On this point, there is some disagreement among the authors.

Where would traditional teaching have guided these people, who came pushed by symptoms and images of the severe father and the search for knowledge, reproducing afflictive fuzziness from this same subject of knowledge? Certainly, it would have guided them to one of those "recycling courses", so that their demands would have been attended to before the image of the severe father, disguised as a teacher, who attempts to transmit knowledge by way of explanations, showing himself as a plain subject. Simultaneously, the symptoms of affliction and detachment would be driven away by a loving hoax and by whatever trick at the hour of evaluation, always in the hope of, by having learned, no one would demonstrate fuzziness nor affliction. To the degree to which traditional teaching repeats this scheme, there is a symptom there as well, that calls for interpretation. As a repetition, it has to do with *juissance*, but whose?

Instead of this, we followed the opposite path. Instead of trying to suppress or modify the symptom, we strove for people to identify with it. Interpretation of the symptoms (Cf. Szizek, 1992, Cap.VII).

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When change becomes the name of the game:

Mathematics teachers in transition to a new learning environment

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The paper presents some of the data collected during the first stage of an ongoing study on secondary-school mathematics teachers in the process of transition to a new learning environment. The study has been inspired by the first author's own experience documented in the journal she kept during the first year of her participation in the innovative project. In the paper, the focus is on two other teachers from the same school, one of them with twenty years of experience and the other one just beginning her professional career. Their attitudes toward the change, which are presented and compared below, have been investigated with the help of a written questionnaire and tape-recorded interviews.

Introduction

The idea of the study presented in this paper was born out of the experience of the first author, Ilana, through which she went when an innovative mathematics curriculum was introduced to her school. The project, called "Seeing Mathematics", promoted an intensive use of computers and was based on group work and guided inquiry learning (Yerushalmy, Chazan & Gordon 1988). For an experienced teacher like herself, with twenty five years of traditional frontal teaching behind her, this meant a true professional earthquake. She was thrown all at once, after only a short period of general preparation, into a completely new learning environment in which everything -- the students and their learning, the teacher and her role in the process, and even the mathematics itself -- seemed completely different from what she was used to. She felt that her past and new classrooms were words apart. No wonder, therefore, that she joined the project with lots of doubts and fears. As a reflective practitioner, and in accordance with the advice she had found in recent literature on teacher change (e.g. Tripp, 1987), she decided to document her experience in a personal journal (Levenberg, 1995). At first, the journal served as means for "blowing steam" and for making explicit the many dilemmas she was facing day after day. Along the way she found, however, that being a mirror of what was going on in the classroom, the journal did not always please her. More often than not, it just reflected her lack of satisfaction

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with the situation, with students' learning, with her own teaching. There were so many things she struggled with, so many changes she had to adjust herself to, and so many new things to learn.

While re-reading the journal by the end of the school year she felt that in order to come to grips with what she went through it might be helpful to compare her own experience to that of other teachers. Thus, she was happy to begin the comprehensive study on teachers in the process of transition to new learning environments, which is now under way. A few initial findings of the first stage of the project will now be presented and cast against the background of excerpts from Ilana's journal.

Although in the paper we will only refer to a very small initial segment of the study, a few words about the whole project would be in place. The overall aim of the investigation is to document and analyze the *process* of teacher's change rather that just its outcomes. Of special interest are the ways in which experienced teachers who find themselves in a new learning environment reform their *beliefs* and *knowledge* (compare Shiffter, 1994; Russell et al, 1994) on the one hand, and adjust their *teaching practices*, on the other hand. A number of case studies conducted during one year of teaching in the new environment will be reinforced with data on teachers' attitudes toward change collected within a much larger sample. To get as exhaustive a picture as possible, we have addressed a highly heterogeneous group of teachers, covering a wide range of ages and experiences. The participants were asked to react to a long list of statements, all of them inspired by Ilana's journal (see examples in Figure 1). The same questionnaire was administered to the case study teachers at the beginning of the school year and it will be answered by them again by the end of the year. In this paper we will report on the attitudes of two case study teachers for whom the year of our study was the second year of teaching within the new environment. The data to be presented in the next sections come from the questionnaire and from interviews conducted at the beginning of the year.

Case study teachers: Dafna and Irit

Dafna[1] is an experienced teacher whose professional biography is quite similar to Ilana's. She has been teaching for twenty years now and spent the last eleven

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[1] Case study teachers' names have been changed

Fig. 1: Excerpt from the attitudes questionnaire[*]

ITEM	DAFNA	IRIT
1. The expectations I had from teaching with computer[**] have been fulfilled	4	1
2. I had many fears before I started teaching with computer	1	5
3. I was afraid mainly of		
a. technical problems	2	4
b. of students' superior computer skills	2	5
c. of too high a number of students in the lab	1	4
d. of disciplinary problems	2	4
e. of unexpected mathematical problems	5	5
f. of the amount of work	5	5
g. of not knowing what to do in the lab	5	5
h. of not being able to cover the material	4	5
4. New learning environment significantly changes my role as a teacher	5	2
5. In the new learning environment I have to prepare myself to lessons much more carefully	5	5
6. In the new environment the ways of assessing students' work change completely	4	4
7. The computer significantly improves the teaching	4	2

[*] 1 strong agreement; 2 agreement, 3 neutral position; 4 disagreement, 5 strong disagreement

[**] This expression, as well as "learning with computer" or just "the computer" (see item 7), are but metonymies for the comprehensive change in the learning environment, the change which includes transition to inquiry learning and group work

years in the comprehensive secondary school in which she is now working. Like Ilana, she has been teaching mainly at the senior secondary level, but she does have some experience with the younger students. As a participant of the project aimed at the junior secondary level project, she is now teaching in the seventh grade (students' age: 12-13). Until eighteen months ago she has never tried any of the innovative forms of teaching on which the new curriculum is based. Thus, even now, after one full year in the project, she may still count as a newcomer to the environment where mathematics is being done with computers, where learning occurs through inquiry, and where students work in groups while the teacher is expected to act mainly as a guide in problem-solving and as a facilitator in classroom discussions.

Dafna's younger colleague *Irit* is a new and relatively inexperienced teacher. The year of our study is the third year of her teaching career. She has got her professional preparation at Haifa University were a thorough introduction to the "Seeing mathematics" curriculum and to its underlying principles is a regular part of pre-service teacher training program. Thus, what for Ilana and Dafna was new and foreign, for Irit was familiar and natural. As a teacher, she has already been born into the new environment. Since, however, during her first year in the school the new curriculum has not yet been introduced, she spent that year giving traditional frontal lessons.

Last year, Ilana, Dafna, and Irit joined the innovative project as seventh-grade teachers. Along with all the other teachers in the school involved in the experimental teaching, they participated throughout the year in a series of project meetings. Although at the face of it the meetings provided a perfect opportunity for sharing experiences and for mutual support, they were devoted exclusively to planning further instruction. For Ilana, her journal was an only outlet for her feelings and doubts. She has not shared her notes with anybody and until the beginning of the present study she has never asked her colleagues about their experience. It is worth mentioning that Irit, notwithstanding her superior preparation, was an ardent -- and widely accepted -- participant of the project team meetings.

In the reminder of this paper we shall quote from Ilana's journal and then describe selected aspects of the initial attitudes toward the change reported to us by Dafna and Irit. We should stress here that in the situation of a radical change in learning environment, teachers' own testimonies turn out to be a rich source of valuable information. As evidenced by Ilana's journal, the change boosts teacher's reflectivity, and raises the level of her awareness and self-awareness. When she can no longer rely on her old instincts, the teacher must make a conscious effort to regain the lost balance. While struggling for adjustment, she becomes much more aware of what is going on around her. So, for example, Ilana's eyes suddenly opened to students' difficulties and lack of understanding:

> *There is a huge gap between my understanding and that of the students. I can feel it clearly ... Only now, after so many years of teaching, do I discover the big difference between what I would like them to understand and what they actually understand.*

She has also become much more self-aware -- not necessarily a blessing for her self-confidence:

> *In many ways, I feel as I used to in my first year as a teacher, except that now I am much more critical toward myself. In the first year I was quite pleased with myself, as far as I remember... Why is it that as a young teacher I did not have all these doubts? I just jumped into the water and swam.*

On expectations and fears

In her journal, Ilana does not mention any expectations she might have had while entering the project. Instead, she meticulously documents the numerous difficulties she encountered and the many fears she had to live with throughout her first year of the innovative teaching. After a few weeks in the project, she wrote:

> *The technical aspects still take much too much time. The lack of technical skills bothers me a lot. I would like it to flow just like writing on the blackboard, but it is not so... I still cannot relax and just peacefully move among the tables and computers... I am tense all the time, preoccupied with urgent technical problems: the mouse does not work, the computer refuses to upload a program.*

Ilana finds it particularly difficult to cope with the uncertainties and the lack of clear structure inherent in the new ways of teaching:

> *Here* [in the project class], *I am devoting incomparably more time to preparations than I use to in my other classes. In the other classes I can plan everything and then I am confident the lesson will not be bad. Here, uncertainty lurks from every corner... At the next meeting we are going to prepare a new worksheet. I always feel more comfortable when the things are structured and ready. Both I and the kids feel it is easier to work this way (old habits die hard?)*

As can be judged from their reaction to item 1 in the questionnaire (see Figure 1), Dafna and Irit came to the project with differing expectations. In fact, Dafna had hardly any expectations at all:

> *I had no special curiosity because I didn't know where I was heading to. I didn't really have any one to ask... I knew I had to face it, and that's it.*

Irit, on the other hand, was full of anticipation. Recalling her one year of frontal teaching she said:

> *I was really waiting for the day when I would start teaching differently from how I was taught in school.*

Above all, she believed that the new ways of teaching will greatly improve students' understanding of mathematics:

> *I see* [the program as providing] *a wonderful opportunity to show that math is one whole and not a collection of separate subjects. Here I can move back and forth all the time. 'Seeing Mathematics' makes understanding much easier.*

The difference between Irit and Dafna grows even bigger when one compares their answers to the questions on the fears they had while entering the project (items 2 and 3 in Figure 1). While Dafna testifies to being scared of great many things -- technical problems, students' superiority, disciplinary problems, etc., Irit declares her readiness to cope with whatever difficulty she might encounter.

In the light of this, Dafna's lack of enthusiasm about the project becomes quite understandable:

> *When I started the new program I knew nothing, I was scared to death. All the technical skills were difficult for me. At the first stage of adaptation, learning to control the computer is the main difficulty. All the efforts and thoughts are focused on that, and there is also the fear' of course, that the students will be better than you are.*

On the change in the ways of teaching and in teacher's role

How the teachers perceive the nature and significance of the change in the learning environment on the one hand, and what they think about the ways in which their own practices should be modified, on the other hand, are two related but different questions. Ilana's notes make it clear that she views the change in the learning environment as quite far-fetched, almost revolutionary. In the new setting, everything seems different: the way the students work in the lab ("*Forty students sitting in pairs and talking*"), the way they act back in their classroom ("*Mathematical discussion is a new phenomenon*"), the pace of the learning ("*A sense of waste of time would not go away; I keep thinking about how much I could manage to do in the regular classroom*"), the methods of assessment ("*Lab reports are not easy to manage*", "*In the tests there is more than one answer to every question... there is no possibility of copying from other students*"), and the list is still long.

On many occasions, she reports on the difficulties she has with adjusting herself to the new scene and reflects on this difficulty's possible reasons:

> *I have habits and expectations as to how the classroom should look like. Evidently, these habits are very difficult to change. Forty students sitting in pairs and talking one to the other! Do they learn? Yes, it seems so. The computer is an exciting tool and in many respects it is more successful than me and my talking. But one has to know to get used to it.*

Elsewhere, she complains about her inability to put up with how the new class *sounds* like:

> *I find it difficult to get used to the noise in the classroom. Arik, the lab assistant, says it is all right but I still can feel the need to control the classroom.*

The inability to give up her central role may be the reason behind Ilana's nostalgia for frontal teaching, expressed time and again in her journal:

> *I prepared a frontal lesson introducing variables. Frontal teaching raises the level of my self-confidence. If I feel students' eyes on me, I know that I am teaching (am I?)*

It should be stressed that Ilana's difficulties do not make her hostile toward the change. Being fully aware of the resilience of old habits, she is prepared to suspend the judgment until she gets a better grasp of the new ideas and until she feels more comfortable with the new practices. She keeps reminding herself that if there is a difficulty, the problem may be basically hers ("*I have not fully adopted yet the idea of inquiry learning*"), and she repeatedly expresses her commitment to the change ("*One has to learn to accept this [e.g. the noise in the lab] and modify her own behaviors*"). Her vision of the change in the environment and of the required change in her own habits are fully consistent with each other.

Dafna's attitude seems less coherent. In the interview, she admits that the change brought to the classroom by the "Seeing Mathematics" project is comprehensive and far-reaching:

> *The entire approach changed, even when I teach without computers I let the students 'explore', arrive at things on their own, I say less.*

On the other hand, her negative response to item 4 in the questionnaire ("New learning environment significantly changes my role as a teacher") shows that in spite of the change in the environment, she believes in the possibility of preserving her old role and old habits.

There is no such inconsistency in Irit's attitude. Like in Ilana's case, her awareness of the significance of the change is accompanied by a conscious effort to be a new kind of teacher. Unlike Ilana, however, Irit does not have to act against her own habits. The only comparisons she can make are with the teachers who taught her in the past:

> *I am not going to teach the same way I was taught In my class pupils get an assignment and start to explore, and this is certainly very different from what I did in school I show students my confidence in their ability to cope. I feel that if I were just standing in class and teaching, their creativity would be lost*

Conclusions and questions for further study

The three teachers we have presented in this paper disclose three quite different attitudes toward far-reaching change in the ways of teaching. Irit views the change brought by the computer and inquiry learning as radical and enters the new environment with obvious enthusiasm. This does not come as surprise. Her ability to appreciate the significance of the transition stems from the fact that while studying the new program at the university she had a chance to get to know it in detail and to make its principles her own. Moreover, being a young teacher with no experience behind her, she does not have deeply rooted habits to change. For her, the new teaching method is the only one she knows from her own teaching practice.

Both Ilana and Dafna are also fully aware of the radical nature of the change in the learning environment. However, while Ilana recognizes the necessity of modifying herself as a teacher, Dafna seems convinced that she nay adhere to her old habits. At this relatively early stage of the project, neither Ilana nor Dafna seem very excited about the change. This lack of enthusiasm on the part of the experienced teacher is easily understandable. There might be much promise in a far-reaching change of the learning environment, but from the point of view of a person who does reasonably well within the traditional setting, the change is also -- and perhaps above all -- a threat. While turning old habits into useless and old instincts into obstacles, the change ruins teacher's self-confidence, forces her to work harder, gives her a feeling that she is competing (and losing!) with younger colleagues or even with her own students, in short -- deprives her of all the advantages of being experienced, the advantages to which she got used through many years of practice. She is left with only disadvantages of her rich experience: with her acute, often paralyzing, awareness of the many difficulties and pitfalls which threaten to hinder the process of teaching and learning.

In this paper, we have presented only the first preliminary fragment of the study, and even here many interesting observations have been left out for the lack of space. While these words are being written, Dafna, Irit, and a few other teachers are regularly interviewed and observed in their classes. Our aim is to describe the process of change they are going through. While doing this, we will try to get hold of those psychological phenomena which seem inherent to the process rather than dependent on the particularities of its external causes. Of special interest to us is the question whether the change of teaching practices keeps pace with theachers' evolving beliefs.

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THE COMPETITION BETWEEN NUMBERS AND STRUCTURE

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In this paper we intend to show that naive solvers do not consider the properties of operations to be context-free -- they perform as if some of these properties are dependent on the specific numerical context. The way they interpret an expression seems to be dependent upon such factors as the place of a particular operation within the expression, and particularly the unique numerical combination at hand. In order to investigate some of these hypotheses we presented 6 grade students with three different numerical versions to the same algebraic structure. The results support our hypothesis that the specific number combination in each example encourages the use of certain sequences of operations and discourages others. The particular number combination in the expression competes with the algebraic structure.

Hidden Structure

Previous research has pointed at structure as a central notion in the study of algebra (Kieran, 1988, 1992, Booth, 1984, 1988) These researchers attributed many of the fundamental difficulties experienced by beginning algebra students to their failure to identify equivalent forms of an algebraic expression according to the properties of the given operations According to Kieran (1988), structural knowledge means being able to identify "all the equivalent forms of the expression" Linchevski and Vinner (1988) argued that this definition should be modified to include the ability to discriminate between the forms relevant to the task--generally one or two forms--and all the others Booth (1981, 1984, 1988) emphasizes that students construct their algebraic notions on the basis of previous experience in arithmetic Thus their algebraic system inherits structural properties associated with the number system they are familiar with She suggests (Booth, 1988) that "the students' difficulties in algebra are in part due to their lack of understanding of various structural notions in arithmetic". Lins (1990), suggests that the students' informal processes, which are probably borrowed from their previous knowledge, are "incompatible" with algebraic methods Thinking algebraically means, according to Lins, operating within the right field of reference Using other sources of reference results in inadequate ways of structuring the situation Greeno (1982) and others (e g Kuchemann, 1981; Avila, Garcia and Rojano, 1990) have found that, to a great extent, these structural difficulties are quite tenacious and tend to last a long time Even repeated exposure to algebraic structure does not eliminate them

Matz (1980) claims that students' mistakes stem from either the use of a known rule without modification in a new situation where it is inappropriate or from the incorrect modification of a known rule in order to use it to solve the given problem She defines the notion of linear decomposition as working with a decomposable object by treating each of its parts independently. She claims that the problems experienced by students who make mistakes in working with algebraic expressions begin when it becomes necessary to abandon linear processing. These students continue to use the old method of decomposing the situation into non-interactive sub-processes, trying to attain the goal by attaining each sub-goal in a linear sequence. According to Matz, the solutions of naive solvers to unfamiliar algebraic problems are uniform and fixed, whether they are correct or incorrect. Even the occasional mistakes made by experienced solvers are surprisingly uniform. This leads her to claim that there are regularities in the way people use familiar rules in unfamiliar contexts, whether successfully or not. There are regularities in the way they perceive the structure of a new problem according to their prototype of a familiar problem or rule, whether correctly or wrongly. These assumptions lead Matz to search for a general theory which considers structure one of the main factors in algebraic competence and to suggest a method of categorizing errors.

Greeno, in contrast, claims that the mistakes related to the mathematical structure are random and inconsistent. In his paper of 1982 he mentions a specific difficulty of beginning algebra students, the fact that they partition algebraic expressions into component parts in ways which seem aimless, mistaken and arbitrary.

Numbers as a cover story

In the present paper we intend to show that the difference between these two views--the one claiming that mistakes are consistent and the one claiming that they are random and unsystematic--is partly the result of ignoring a major element in the process. This element is the context of the expressions--in our case, the numerical context. Our assumption is that at the beginning of algebra the focus shifts from concern with the numerical properties of the terms in the expressions to the properties of the operations. These properties do not change according to the arbitrary, one-time configurations of the numbers being operated upon. This way of looking at the matter leads naturally to viewing expressions as being context-free entities and there-by to the concept of structure as a central one in analyzing the way students work. The legitimate structures of an expression are compared to the structures used by the solvers, in an attempt to either find some systematicity in the solvers' use or to reach the conclusion that there are no regularities in this area.

We will try to show that naive solvers do not consider the properties of operations to be context-free -- they perform as if some of these properties are dependent on the specific numerical context. The way they interpret an expression and the alternative structures that they spontaneously associate with it, seems to be dependent upon such factors as the place of a particular operation within the expression, and particularly the unique numerical combination at hand.

For example, if we compare the expression (1) $217 - 17 + 69$
with the expression (2) $267 - 30 + 30$

we see that the type of numbers in the expressions leads to sequential computation from left to right much more easily in the case of expression (1) than expression (2), as the latter encourages operating on $-30+30$ before operation with the 267. While the first expression will not evoke any alternative scheme, whether correct or wrong, the second expression may evoke the correct alternative scheme of -30+30 "giving zero" (that is, subtracting 30 and then adding 30 is the same as doing nothing). It may also evoke the incorrect alternative of performing 267-60, that is, detaching the 30+30 from the subtraction operation (Herscovics and Linchevski, 1994 ; Linchevski and Herscovics, 1994, Linchevski and Herscovics, 1996) Examples like 530-10+10+10 should greatly increase the probability of evoking this sort of incorrect scheme

We assume that an example like (1) 12 x 5 ÷ 2 x 17
should evoke an entirely different scheme
in some people than the following example (2) 150 x 2 ÷ 2 x 150
with the same algebraic structure

It is quite plausible that the correct solution--computation from left to right--will be chosen most often in the case of (1), while there will be a tendency to choose alternative schemes, which are not necessarily correct, in the case of (2).

A similar contrast should obtain with the following two
examples (3) 100 ÷ 2 x 25 ÷ 5
(4) 25 ÷ 5 x 2 ÷ 7

Whereas in the first pair [(1)&(2)] the different schemes will sometimes lead to a correct solution and sometimes to an incorrect one, in the second pair [(3)&(4)] the ultimate result will not necessarily reveal the use of alternative schemes. It is likely that a student who solved 100 ÷ 2 x 25 ÷ 5 by inserting "mental parentheses", thus actually calculating (100÷2) x (25÷5), which gave 50x5=250, was not doing this for the right structural reasons, out of a flexible view of the algebraic structure: a÷bxc÷d = (a÷b)xc÷d = (a/b)xc÷d = [(axc)/b]÷d = (axc/b)x(1/d) = (axc)/(bxd) = (a/b)x(c/d).

It's more likely that the particular numbers that appeared in this example were what led the student to the view according to which the calculation was performed.

We therefore decided to investigate some of these hypotheses. In this paper we will report on some of the examples we designed and administered.

Method

In order to sample a wide range of abilities, we interviewed individually all sixth graders in two classes of the public school system in Israel and in Montreal, where the average age of the students interviewed was 11:6. All the students had learned the order of operations in class prior to the interview. Each subject was interviewed individually. We prepared a script with exact wording of each question. However, the interviews were only semi-standardized, in the sense that the interviewer could re-phrase the questions. In our report we present the wording of the questions because we found that even small variations could alter the students' responses. An observer was presented at each interview to take notes, and also participated later in the analysis of the students' responses

The rationale and design of the tasks

Each of the examples was constructed according to the following considerations. Three different numerical versions of each algebraic structure were prepared. In one version the numerical context "went together with" the structure--that is, according to our hypothesis, the numerical combination encouraged operating according to the correct algebraic structure. In the second version the numerical context "went against" the structure--that is, according to our hypothesis, the numerical combination encouraged operating in a way that was opposed to the correct algebraic structure. In the third version the numerical context of the structure was "neutral"--that is, the numerical combination, in our view, did not encourage the solver to operate in either one of these ways.

Consider the following example of an algebraic structure of the form: a - b x c
The following numerical combination "goes together with" the structure: 21 - 2 x 5
The following numerical combination "goes against" the structure: 27 - 7 x 15
A "neutral" numerical combination could take the following form: 20 - 5 x 3

as the two possibilities, doing the multiplication first (5x3) or doing the subtraction first (20-5) are equally appealing.

A different sort of "neutral" combination is illustrated by 131 - 17 x 6 where both possibilities are equally unappealing.

The examples were presented to the students in random order rather than blocks of examples with the same algebraic structure. Each example was presented separately. In our presentation of the results, we put together the data for each block to make it easier to interpret the results. Since the differences between the results of the students in the two classes was non-significant, we combined the data of both classes for presentation in the table.

Results

The students were first asked: **Here is a string of operations. Can you show me a quick way to find the answer to this problem? ... You may use the calculator if you want,** (A simple calculator was available to be used as a number-facts table when needed)

Item 1

	1a) 23 - 5 x 21	1b) 27 - 7 x 51	1c) 27 - 3 x 5
Multiplication first	61%	33%	52%
Subtraction first	39%	67%	48%

Item 2

	2a) 27 - 7 + 3	2b) 25 - 7 + 3	2c) 27 - 5 + 3
Subtraction first	87%	69%	78%
Addition first	13%	31%	22%

Item 3

	3a) 24 3 x 5	3b) 240 15 x 2	3c) 24 3 x 2
Division first	91%	62%	67%
Multiplication first	8%	38%	33%

Item 4

	4a) 8 x 5 2 x 17	4b) 150 x 2 2 x 50	4c) 19 x 16 15 x 17
Left to right	92%	81%	87%
(axb) (cxd)	8%	15%	13%

Item 5

	5a) 75 25 3	5b) 75 9 3	5c) 64 8 4
Left to right	89%	42%	41%
a (b c)	5%	58%	59%

Item 6

	6a) 168 - 20 + 10 + 30	6b) 130 - 10 + 10 + 10
Left to right	63%	52%
a-(b+c+d)	30%	42%

Item 7

	7a) 147 - 16 4 + 3 x 5	7b) 37 - 5 2 + 4 x 3	7c) 28 - 8 4 + 3 x 2
a-(b c)+(dxe)	89%	74%	89%
(a-b) c+(dxe)	0%	16%	0%

Discussion

The results support our hypothesis that the specific number combination in each example encourages the use of certain sequences of operations and discourages others The particular number combination in the expression competes with the algebraic structure

The algebraic structure is supposed to focus attention on the rules of operations, while the specific number combination is in the best case supposed to encourage searching for the best alternative among the legitimate structures--the one which takes advantage of the specific given number combination--without changing the meaning of the expression The results, however, indicate a somewhat different picture The specific number combination often shifts the focus of attention from the structure to the numerical properties of the given elements in such a way that the meaning of the expression is changed

The interaction between the structure and the specific number combination seems to us to explain, at least partially, what Greeno (1982) sees as random and inconsistent mistakes Greeno claims that the specific difficulty confronting beginning algebra students is that they partition algebraic expressions into component parts in a way which seems aimless, mistaken and arbitrary This difficulty stems at least in part from the competition between the structure and the biasing number combinations, which frequently encourage decomposition of the expression at the wrong points.

This phenomenon should not be surprising. In other areas, such as logic and language, there are many instances of sentences with the same logical structure conveying different meanings when they have different linguistic content. Earlier research of Linchevski and Nesher (1978) investigated students' logical judgment of sentences with the same logical structure in different contexts. Even after the subjects had studied the truth tables and practiced analyzing sentences and propositions according to the criteria of their logical structure, the verbal context still had a considerable biasing effect on their perception of the logical structure of the sentence and the truth value that should be assigned to it.

Leonard (1994) makes use of the distinction between the metaphoric and the metonymic axis originated by Sassure. Specific words in a sentence can be replaced on the "metaphoric axis" without changing the logical structure of the sentence, while changes along the "metonymic axis" lead to a change in the structure of the sentences even if its key words remain the same. In the second case it is obvious that the change generally produces a change in the meaning of the sentence, but Leonard notes that even in the first case, when the structure of the sentence has not changed at all, substituting words in this axis has the effect of changing the interpretation.

Is there a conflict between number-sense and structure-sense

Another important consideration is the possible role of focusing on number sense in the competition between the structure of the mathematical sentence and the specific numbers composing it. There may be an interaction between the activities aimed at developing number sense and the great influence of the context on the method, whether correct or incorrect, that students use in working out arithmetical expressions. One of the components of number sense, according to Markovitz and Sowder (1994), is mental calculation, which, they claim, involves the invention of nonstandard methods of calculations based on the properties of the numbers. They argue that number sense involves using numbers flexibly thus, in developing number sense the attention should be focused on the **particular numbers** involved in the calculation.

All this means that the bulk of the teacher's/student's attention in activities intended to develop number sense is devoted to the specific numbers used in the structure. This factor is the primary justification for replacing the standard algorithmic procedure by an alternative one. The goal of developing number sense in the student makes it necessary for the teacher to carefully choose the specific numbers and operations to use in expressions presented to the students, so that non-standard solution procedures will lead to more elegant and efficient calculations. In other words, developing number sense involves taking advantage of the properties of the numbers in the specific example.

Moreover, number sense is developed in those educational stages in which considerations of algebraic structure are still only a secondary aspect of the learning process. The examples presented to students are intended to provoke a search for numerical relations between the elements and encourage taking advantage of these relations. We call examples of this sort "context-proof" or "structure-proof" combinations of numbers and operations, as they are designed, intentionally or unintentionally, to avoid any conflict between the immediate, spontaneous alternative procedure, governed by the type of numbers involved and the algebraic structure. In our opinion, examples like these should be **balanced** by others in which such a conflict does arise, in order to provide an opportunity to illustrate the competition between the structure and the properties of the numbers in the specific combination.

Consider, for instance, one of the examples presented by Markovitz and Sowder, 76 + 53 + 17 - 53. It is plausible to assume (although there is not enough information in the paper) that canceling the 53s was considered as a strategy indicating greater number sense. However, according to Linchevski and Herscovics (1994), the slightly different combination, 76 + 17 - 53 + 53 would have led some of the students to a mistaken procedure. They would have added the 53s as if the combination had been 76 + 17 - (53 + 53), due to what have been called "detachment of a term from the minus sign" (Linchevski and Herscovics, 1994).

To be sure, we do not claim that flexible, specific solving procedures should be discouraged. They should indeed be encouraged, but not under conditions that could lead to over-generalizations. If they, consciously or unconsciously, turn into a goal in itself they might have dangerous potential in cases where there is a conflict between the structure and what seems to be a correct use of the properties of specific numbers. The danger is that in these cases the numbers will beat the structure in the competition between them.

Following our assessment study we initiated several discussions with the students. The purpose of the discussions was the search for activities aimed at developing structure-sense. We presented the students with conflicting expressions that were very likely to induce a high rate of misinterpretations of the algebraic structure. For example:

D is presented with the exercise: 136 -36 + 29, he solves it from left to right and gives 129 as an answer. The following exercise was: 54 - 2x8 + 20,

D: First I have to do the multiplication, 2x8=16, 16+20 gives 36, 54 - 36 equals 18.

T: I have a question. In the previous question, (136-36+29), another student did the addition first, Is it O.K.?

D: No, it doesn't make any sense, we better go from left to right

T: In the second example you first added 16+20

D: Here it's O.K. because here it does not make any difference. I mean here it doesn't make any difference if I first add 16 and 20 and only then subtract it from 54

T: I don't follow, so why in the first one it is not allowed?

D: Here, (in 136-36+29), I can't start in the middle, but here, (54 - 2x8 + 20), I am any way already in the middle so I just continue, I can do it like that.

We are trying, now, as well, to arise some conflict and to initiate a discussion modeling verbal situations. For example:

Y (one of the students who evaluated 430 - 15 + 15 as 430 - 30) was told the following story: R had 47 marbles. He played two games, in the first game he lost 13 marbles, in the second one he gained 21. Which of the following expressions reflect/s (represents/embodies) the

story... which of the following expressions reflect/s the number of marbles he ended up with ...
(1) 47 -13 + 21, (2) 47 - (13 + 21)

Y: I think... the second one..., no, ... I don't know

T: .... you are not sure...

Y: I'll try (Y starts to evaluate the second string) ...it gives 33, not 34...um...47 minus 34 gives 13....(Y looks very doubtful, not convinced by the result)

T: ...well...

Y: It doesn't make sense

T: Why?

Y: He should be left with more than 47 marbles. I had 47 marbles, I lost 13 and after that I gained 21, I got back the 13 I had lost so I had again 47 and I got even more. I'll try the first one

It seems that viewing the expression as modeling the situation has been a good starting point for a discussion between Y and the teacher regarding the alternative structures of the expressions.

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SITUATED INTUITIONS, CONCRETE MANIPULATIONS AND THE CONSTRUCTION OF MATHEMATICAL CONCEPTS: THE CASE OF INTEGERS

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Abstract

A teaching experiment is described which draws on the children's intuitive understanding of a disco situation and their manipulations of a double abacus to keep track of movements in and out of the disco-gates, to facilitate their construction of the integers and to develop strategies for performing integer operations. The aim is to provide a situation where the children naturally construct the integer concept from the class of gates with the same 'report', and to provide intuitive support for the abacus manipulations required for the integer operations of addition, and crucially, for subtraction. There is evidence that the children can do this. At least certain aspects become 'obvious' to the children even though we have not yet achieved 'comprehensiveness' in the models put forward.

Background

Negative numbers introduce a new aspect into the study of mathematics: for the first time reasoning in an algebraic frame of reference seems to be required. While counting numbers are constructed by abstraction from real objects and quantities, and operations performed on them are related to concrete manipulations, operations on negative numbers and the properties of these numbers are given meaning through formal mathematical reasoning. Moreover, some of these properties contradict intuitions that have been developed in constructing the counting numbers, (for example, you can't get something from nothing!). Over the years this situation has led people in the mathematical community to one of two positions. One alternative has been to completely avoid any attempt to give practical meaning to the negative numbers, and to recommend treating them formally from the outset (Fischbein, 1987; Freudenthal, 1973). The other alternative is to look for an embodiment, a 'model' that will satisfy the need for providing a practical intuitive meaning to negative numbers, arithmetical operations on them, and the relations between them (e.g. Thompson and Dreyfus, 1988; Peled, Munkhopadhyay and Resnick, 1989; Munkhopadhyay, Resnick and Schauble, 1990; Liebeck, 1990; Janvier, 1985).

Fischbein (1987) argues against the use of the existing models for negative numbers. At best such models justify only some of the algebraic properties of these numbers, they do not satisfy the criterion of 'comprehensiveness' (see as well Battista, 1983). Moreover, Fischbein claims that the models are based on artificial conventions and thus do not address the cognitive obstacles confronting the students. He believes that the purpose of a model is to add 'obviousness' and 'correctness' to mathematical concepts and operations on them, but that this purpose is not achieved by the artificial models in current use. Moreover, the very definition of the negative numbers makes it impossible for there to be such a model, because these objects cannot be described directly and realistically. Their existence and the relations among them can only be deduced formally. Fischbein therefore concludes that the topic of negative numbers should be taught only when the students are ready to cope with intra-mathematical considerations and justifications, using 'at least' the inductive-extrapolation method (Freudenthal, 1973, p. 281). Let us now consider the three requirements suggested by Fischbein: comprehensiveness, correctness and obviousness. The requirement that a single model should satisfy the need for comprehensiveness in teaching a mathematical concept is practically impossible to fulfil. Rejecting models because they are only partial would lead to rejection of all the existing models in mathematics education, since by definition every model has aspects that are not in the concept and vice versa (Ost, 1987). Bher et al. (1983, p. 102), note that "the rational number project has shifted away from attempting to identify the "best" manipulative aid for illustrating all rational-number concepts toward the realization that different materials are useful for modeling different rational-number subconstructs."

Moreover, people attempting to solve mathematical problems often make use of several models in the process of finding the solution. Different parts of the problem may lend themselves to the use of different representations, including the combination of concrete thinking with abstract formal reasoning (Bher et al., 1983; Usiskin, 1988; Sfard and Linchevski, 1994; Gray and Tall, 1994).

The requirement of correctness in models is especially interesting. Resnick and Ford (1981) also claims that the main purpose of a model is to create a mental image of 'goodness' and 'correctness' for the system of concepts being learned. According to these views the purpose of a model is not merely to provide a well-defined interpretation for a mathematical theory but also to give the theory or concept a 'correct representation'. This cannot possibly be fulfilled. Every mathematical theory has or can be given alternative models that provide the user with different images of the concepts in the theory and the relations among these concepts. Fischbein's requirement of correctness stems from the fact that the new concepts being acquired are often extensions of existing concepts (Semadeni, 1984). Therefore the proposed model must preserve the intuitions and schemes that were constructed in the narrower frame and transfer them to the extension. When this condition is satisfied, the person using the model has a feeling of 'correctness'; if it is not satisfied, the person has a feeling of 'fabrication' or 'obscurity'.

Inherent in the 'obviousness' criterion is the requirement to avoid artificial conventions that would make a model seem detached from reality. Moreover, in order for the model to fulfil its cognitive function it must describe a reality that is meaningful to the student, in which the extended world (for example, the world which contains negative numbers) already exists and our mathematical activities allow us to discover it (e.g. Vinner, 1975). In the specific case of negative numbers this world must include the practical need for two sorts of numbers. It is also necessary to present situations in this world in which the relevant laws can be deduced without 'mental acrobatics' (Janvier, 1985), and without inducing a feeling of contradiction with known truths.

In this report we describe an experiment in teaching the negative integers to sixth-grade students, with an attempt to fulfil the third of Fischbein's (1987) criteria, that of 'obviousness' for addition and subtraction of integers. The construction of the integers essentially involves the construction of an equivalence class of pairs of natural numbers, involving a recognition of the 'sameness' of a class of pairs such as {(5,0), (6,1), (7,2)...} and the attachment of some label or sign, eventually this will of course be +5. We want this to be constructed intuitively thus, the 'procept' (Gray and Tall, 1994), for the integer will attach itself to an action-in-situation (which holds some meaning and can evoke intuition), a representation on an abacus (which can be manipulated independently) and some label, initially just a verbalisation "5 more in", but which in a later episode becomes the formal mathematical symbol, "plus 5".

Design of the study

The study involved a series of teaching episodes with small groups (of three children at a time usually) of year 6 pupils who have not yet received any instruction in negative numbers. The researchers are both teachers and interviewers, actually participant-observers who presented the whole exercise to the children as an activity in which they will help to improve the teaching of other children later. All the meetings were videotaped to allow further analysis. The sequence of episodes is designed to lead the children to construct the integers from their experience with a game while using an abacus as a manipulative aid. The teaching interventions are designed to identify opportunities and obstacles the children meet on the way, the role of their intuition for the situation (situated intuition), their use of the double abacus and the spontaneous strategies they develop.

A disco situation (D) is presented which is designed to embody the ring of integers under addition as an extension to addition and subtraction of natural numbers. This is done in the context of a crowd of people at a disco, with more disco dancers arriving or leaving every minute by a number of gates. Each gate is controlled by a child, who is asked to keep track of the numbers coming in and out, (these being the embodiment of the natural numbers to be extended) using an abacus (A) with two wires and two colours of beads (yellow and blue) labelled OUT and IN, respectively. The 'game' is played by the children drawing cards which symbolise (S) the number of dancers arriving (blue card, later a pink card with a plus sign, with a numeral 0,1,2,3,4...) or departing (a yellow card, later pink card with a minus sign, with a numeral, 1,2,3,4...) Occasionally the children are required to report the action at their gate to a controller who records the total scores on the controller's abacus, so as to decide if the room is getting dangerously full. A rule is made up that if the controller records 20 more dancers in the room than when the game started, the disco closes and game ends. (In later sessions these recordings entered the game as instructions on cards to "Report to controller: if the room is too full then stop"). The children are encouraged to make sense and to extend their understandings by translation between the three elements of the triple: D, A, and S.

Results: recording and reporting, towards the notion of equivalence

The children take to the game, the symbols on the cards and the recording on the abacus naturally: the disco-game seems 'real' for them. In the first sessions, the video and transcript show that the children find it obvious to record the movements of people in and out of the gates with the double abacus, and to report the change in the numbers in the room according to the difference in the piles of beads on their abacus, thus the children (Je, A and S) report to the teacher (T):

Je: *One more..... There is one more on this side than on this side.*

T: *How can you tell.*

Je: *That one is higher than this one........... One.*

T: *Why are you sure?.. Did you count or there is a quick way?*

(Je looks at the abacus checking with her eyes only)

T: (to A) *What about you , A?*

A: *Sure.*

T: *Show us.*

(A moves the extra blue bead up the rod with her finger to show the extra one.)

The natural reporting by the children of the change at their gate is performed either by a) counting all the blues/ins and the yellows/outs and taking the difference, or b) identifying the difference between the piles and counting them. For small differences it appears to be a visual subitisation, for larger numbers the children tend to raise the beads which signify the difference and then count them. A third strategy is to take away the same number of blues as yellows from both sides to obtain a simplification:

T: *Stop we might be full. ....* (to S) *What are you going to report?*

S: *More, one.*

T: *So can you cancel out so you have only one on your abacus?*

(S pushes all the yellows and all the blues besides one blue to the back of the abacus.)

T: *What did you just take away?*

S: *Oho, goodness!!*

(He does not know how many beads he has removed from each wire, which shows that he wasn't counting the beads he took away, he noticed only the difference.)

T: *Does it matter?*

S: *No!*

It seems here that the child attended to the difference between the piles of beads and reduced the gate to a simple equivalent form: this has only rarely arisen spontaneously so far, but can arise after teaching about equivalent gates. When asked to produce a gate with the same report as one shown to them, but maybe busier or quieter, the children, without any difficulty, can produce busy gates and quiet gates. Usually the 'quietest' gate produced has at least one bead of each colour, but in most cases the canonical gate (with beads of only one colour) is eventually produced as the 'quietest possible gate'. However, when asked to reflect on the transformations they make to obtain these equivalent gates, the children may still attend to the difference between the piles of beads, and maintain this difference (the 'three more blues') by placing arbitrary numbers of both yellows and blues and then adding the required difference 'of three blues'. They do not necessarily attend to the addition and subtraction of equal numbers of beads as such, though this may be spread from child to child or may require focussed teaching.

Towards a concept: flexibility in selecting a representation

Another obstacle arose when the children played the game on until they ran out of beads. A variety of solutions have been offered by different groups. Some want to stop the game and report to the controller. One ingenious solution was to send the people to another gate which still had some beads (though the children saw this wasn't realistic in the situation, and therefore that the game should outlaw this). In the following extract we see a group which first chooses to record the extra movements in writing, and then develops 'cancelling', 'cancelling-elision' and 'compensation' strategies. This is in their second session of the game, after some work the previous day on reporting, equivalent gates and their canonical forms, and some spontaneous recording in writing when the disco became too full. The game progresses until the abacuses become nearly full, then Ad and Lu suddenly have too few beads available to them to record the cards they have picked up. Ad has a blue-2-in but only one blue left, and Lu has a blue-4-in card and only one blue bead left to bring over (she also has three yellow beads available to bring over). Lu suggests writing it down as in the previous session, and the teacher presses for an alternative solution: "Is there another way?"

Ad: Ah . . . you take one off . . . there. .

Ad is removing a yellow bead from his abacus, a compensation for adding a blue, checking with the teacher non-verbally to see if this is OK. But he is not certain enough on his own and T's prompt "go on' isn't enough. Lu interrupts, she takes her one blue, and not having the other three blue beads she decides to cancel 15 beads from each of the wires. This is done by taking back 15 yellows and 15 blues from the front of the abacus: but she counts the 15 blues starting from 3, thus leaving the three blues which remained to be recorded!).

Lu: You take 15 off there, (15 yellow beads go over) then take 15. ..(She means blues but she starts counting from 3), that 'cos I added the 4, ..I added 3 on because of the 4 ...take 15.

T: VERY interesting, now what did you do there?

Lu: Well the 15 people that had went out, I took off the 'ins' and so I pushed all . . .

Here she justifies her manipulation by appealing to the situation of the outs and ins. Notice also the revealing language where the beads are spoken of as 'outs' and 'ins', that is to say they are the objects, nouns, which represent the history of the process of people moving in and out. We now return to Ad's solution to his problem, adopting Lu's method now:

Ad: Could I go like that, 'cos that? (takes 2 off both sides puts 2 blue beads back because of the blue 2 card he is wanting to count.)

T: What has he done there, Cl?

Cl: Well, here, it was, ...had no change, so he took two off, (she means off each wire) *because there'd still be no change,* (she means the report would still be 'no change' even after cancelling two beads from each wire) *but that'd make it the same, but then he wanted two blues so he put them over.*

The game continues, Ad gets a blue 4 card and is again stuck. Because Lu took 15 beads of each colour over she has plenty of beads available and has no problems, Cl got a zero. Ad takes 4 yellows away from his abacus because he has to count a blue 4 card but has not enough blue beads left:

Ad: Is that OK?(Ad pushes back 4 yellows) *Could I do that?* (looks around)*: Is that all right?*

Lu: You should take 4 blues and 4 yellows away but then put the 4 blues back.

In the ensuing game, Cl takes three off both sides so she can put 3 blues back, (cancelling). Lu is cancelling but sometimes is seen clearly to elide the process, half transferring a bead before letting it slip back (cancelling-elision). Ad is using a different strategy: taking off yellow beads instead of adding blues. It is not yet clear if this is cancellation with elision, i.e. he simply curtails the manipulation, or whether he sees the actions as really compensatory.

Ad: Seeing as I've got one 'out' (he means he has a yellow one-card and wants to add a yellow bead to the 'outs' pile, but can't) *..can I take one off the 'in'?* (Here he wants to take away a blue bead instead of adding a yellow. Before, he was always taking off the yellow pile instead of adding to the blue. We may see this as a recognition of the symmetry of these manipulations, but he still has doubt because he lacks justification in the situation of the disco.)

. . ...Because .. um.. This bead has gone 'out' so I put one 'out', and this 1 'out' so I put one on the 'ins'.

Then Ad gets stuck again, he has run out of beads in a new way. Actually he has reached the limits of the abacus. He needs 4 blues and he only has one blue at the back of the abacus, and 2 yellows at the front of the abacus.

Lu (to Ad)*: Take 4, no take two out then put two of the yellow ones in . . . and then take . . .just write it down until you get, . . . write "one in".*

This extract illustrates a number of features, but especially the way the children can develop strategies based on an appreciation of the equivalence of the different gate representations, (they have the same report, the same effect on the numbers in the room, and this justifies cancelling and bringing over the same numbers of yellows and blues when needed). They also justify certain equivalences of operations on the abacus, viz. the equivalence of adding blues and taking away yellows. This lends potentially intuitive support to later algebraic formulations. Notice the importance of the abacus in reifying the processes in the situation. The children are able to manipulate the beads fluently without recalling their situated meaning each time. They begin to think and talk about the actions on the beads themselves, and their equivalences. Finally, we note the language constructed by the children, in referring to the beads as 'ins' and 'outs' illustrates the duality of the process-object of an integer elegantly, and helps them to translate from the disco situation to the abacus and back again as necessary.

Integrating explicit subtraction

We aimed at constructing subtraction naturally from the (disco) situation while taking advantage of the multi-representation of each number, as a natural extension of the concepts acquired through previous experience with the game and manipulations. As long as subtraction meant 'taking away' a number of beads from the relevant wire and there were enough beads on each wire to take off no cognitive obstacle has been raised; e.g. -8 take away -3 is -5, since the -8 and -5 were understood as numbers of yellow beads, and the taking away maintained the same sense as in their previous experience with Natural numbers and sets. However, when calculations involve a passing through the zero, e.g. +3 take away +5, or +3 take away -2, in which the beads are not there to be taken, a shift into the idea of the equivalent reports and representations is required. Subtraction was justified to the children as the operation required to recover the state of the gate before a given card had been added. When presented with the need to undo a card such as -5 when there are only 2 yellow beads on the abacus the children could reason that there must have been some cancelling done, and were able to un-cancel by bringing forward yellow and blue beads sufficient to be able to perform the 'take-away'. This finally led some children to the point of symbolic subtraction using the double abacus to represent the integer and carry out the required manipulation.

Discussion

Following earlier research by Diriks (1984) and others the teaching was based on a model in which the neutralisation of equal amounts of opposites allows every integer to have many physical representations (Lytle, 1994). This feature supplies grounds for constructing addition and subtraction as an extension of the children's existing schemas. However, in contrast to most of these studies we did not want negative integers and the operations on them to be introduced formally in the context of a ready made model accompanied by an imposed set of operating rules. We wanted them to be constructed intuitively so the model developed would be 'obvious'.

The children were encouraged to make sense and to extend their understandings by translation between the three elements of the triple, Disco, Abacus and Symbol. Thus the abacus was introduced as a means of recording the movements of dancers in and out of a gate into the disco, (D--A). The children were asked to report the score on their abacus in terms of "how many more dancers are in the room?' (A--D), and to undo a recording by 'reversing the recording on the abacus' by removing the required number of beads and this was interpreted later as subtraction. Manipulations and simplifications of the abacus were justified by the teacher or students with reference to the meanings in the disco situation (involving A-D, then D--A again). Key elements in these translations were the verbalisations of the children and teacher, which facilitate and describe (but in some cases hinder) the making of the appropriate connections. We have found that the abacus manipulations lent additions and some subtractions immediate 'obviousness'. Subtraction which involved passing through the zero (e.g. -2 take away +5) requires flexible use of abacus representations which are justified by the equivalences of the reports in the game, and in which the obviousness to the children is based on the strength of the connection with the situation in which the intuition is based. Further work on the this strategy is in progress.

The children, as an answer to the abacus' limitations, have spontaneously introduced written symbolism into the 'game'. The introduction of formal symbols adds two more sides to the triangle of translations. "Plus 3" symbolizes the action of adding to the abacus, as well as the resulting report of the abacus "3 more in the room", and calculations can be motivated by situations (D--S), symbolic calculations executed through the abacus manipulations (S-A-S) and new problems solved by using abacus manipulations formalized from earlier justifications based on actions in the disco situation (reification). It seems to us that by the end of the fourth session many of the children were ready to abandon the situation, (D), and to operate directly in the S-A-S translations. However this aspect of the study is also still under investigation.

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SECONDARY PUPILS' TRANSLATIONS OF ALGEBRAIC RELATIONSHIPS INTO EVERYDAY LANGUAGE : A HONG KONG STUDY

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Nearly 600 Secondary pupils in Hong Kong were tested on a series of simple algebraic problems likely to produce the well-documented 'reversal error'. The two problems involving the interpretation of an algebraic relationship and its translation into everyday language were found to have very low facility rates compared to other items on the test. Analysis of these two items revealed clearly identifiable response patterns that were very stable across both problems. These response patterns are illustrated and possible underlying reasons are discussed.

Introduction

The algebraic error known as the 'reversal error' is well illustrated by the following popular example: "In a College there are 6 times as many students as professors. If the number of students is denoted by S and the number of professors by P, write an equation that represents this relation." Typically, the 'reversal error' response is then $6S = P$. The continuing research interest in this phenomenon is due to its frequency, its resilience to correction, and to the difficulty of finding an adequate theoretical explanation. Various hypotheses have been proposed in terms of the misapplication of natural language rules (Kaput, 1987), direct syntactic translation (Mestre, 1988), incorrect frame retrieval (Davies, 1984), and the idea of static comparison (Clement, 1982). More recently, McGregor & Stacey (1993) have argued that none of these models satisfactorily explain *all* the situations in which the reversal error occurs. They in turn propose a theory of cognitive models (developed from Clements' static comparison) in which reversals are explained as "direct representations of cognitive models in which the numeral is associated with the larger variable" (p.228). The present study focused on four situations (and variants) that are likely to produce the reversal error. These are illustrated by the matrix below (non-referent means that no real-life objects are mentioned, simply numbers).

	Multiplicative	Additive
Concrete-referent		
Non-referent		

The Student/Professors problem is an example of the Multiplicative/Concrete case. An example of an Additive/Non-referent case is "p and q are numbers. p is 6 more than q. Write an equation that describes this sentence". The typical 'reversal' response is then $p + 6 = q$. The detailed results of this study are discussed in Lopez-Real (1995). The conclusions supported the McGregor-Stacey theory but also provided strong evidence that the construction of cognitive models is a function of the syntactic structure and lexical items of a given statement as well as its semantic content.

Translation from Symbols to Everyday Language

In common with previous research on the reversal error, the present study concentrated on items in which the pupils had to *construct* a mathematical equation from a statement written in everyday language. (These are referred to as 'construction' items later in this paper). However, two questions were included that required the opposite process; that is, the *interpretation* of a mathematical equation into everyday language. One example of this type of question was used by McGregor & Stacey (1991) at the 'trial' stage of their project but none were included in the final testing. The purpose of including these two items was to determine to what extent the reversal error would still be evident when interpreting an algebraic equation. Since these questions involved interpretation rather than construction, it was anticipated that they would be easier than the other items on the test. This would be in accord with general theories of language acquisition in which 'decoding' precedes 'encoding' and is therefore assumed to be easier (e.g. Aitchison, 1989). In fact, the results showed exactly the opposite with these two questions having the lowest facility rates in the test. Moreover, being more open-ended in nature, it is perhaps not surprising that a greater *variety* of responses was produced. The range and complexity of these responses meant that they could not be adequately covered in the initial report of this study (Lopez-Real, 1995) and were therefore omitted at that time. This paper now deals largely with the analysis of these particular items.

Sample and Background

A total of 577 Form 2 pupils from 6 secondary schools in Hong Kong were tested for the study. Hong Kong has a selective system at the transition stage from Primary to Secondary schools and pupils are assigned to one of 5 band-levels according to their academic performance at the end of primary education. Secondary schools will then cater for a narrow range within this band structure (e.g. a school may be described as a Band 2 & 3 school etc). Schools may opt to use English or Cantonese as the medium of instruction. At the present time about 70% of the secondary schools are *described* as operating an English-medium policy. However, in the case of mathematics it is very common for teachers to use a mixed-code presentation whereby many of the explanations are given in Cantonese with key phrases, especially those involving technical terms, emphasised in English. Nevertheless, the pupils in such schools use English language textbooks and take their examinations and tests in English. The schools chosen for this study were English-medium schools covering Bands 1 to 3. Hence the pupils were 'average-to-high' in terms of academic performance. The test consisted of 10 items.

Items and Response Patterns

For simplicity, the two 'interpretation' items are numbered Q1 and Q2 below, although they did not appear in these positions in the test :

Q1. **Mrs Chang has N oranges and M apples. Write a sentence to explain the meaning of the following statement: N = 3M.**

Q2. **Mrs Leung has P oranges and Q apples. Write a sentence to explain the meaning of the following statement: P = Q + 5.**

Nine clearly distinguishable response patterns were identified for both the additive and multiplicative problems. Idiosyncratic or rare responses were classified as miscellaneous and there was a final category for omissions, making eleven in all. Each of the nine response patterns is listed below with a brief explanation and illustrative examples.

i) Correct Description

The most important characteristic that had to be evident here was some understanding that the equations referred to the *number* of oranges and apples. Nevertheless, very flexible criteria were used as far as grammatical construction was concerned. All of the following were admitted:

The number of oranges is 3 times of apples. (Q1)
Mrs Chang's oranges are 3 times as much as her apples. (Q1)
Amount of oranges is 5 more than apples. (Q2)

ii) Reversal Description

This involved a sentence structure similar to (i) but with the relationship reversed:

Mrs Chang has 3 times as many apples as oranges. (Q1)
She has 5 more apples than oranges. (Q2)

iii) Implied Reversal

Here a 'ratio' or 'unitary' description was given, or an illustration using numbers, that implied a 'reversal' conception:

There are 3 apples for each orange. (Q1)
If she has 2 oranges she has 2+5=7 apples. (Q2)

iv) Direct Substitution

A description using both fruit and the letters of the equation that suggested pupils simply substituted words into the equation :

N oranges is the same as 3M apples. (Q1)
P oranges is equal to Q apples add 5. (Q2)

v) Correct Relation

A sentence using fruit and *one* of the letters in the equation. This is in contrast to (iv) insomuch as the statement now suggests that the correct *relationship* is understood:

If she has M apples she has 3M oranges. (Q1)
There are P oranges and P-5 apples. (Q2)

vi) Number Relation

This is characterised by the fact that *no* reference is made to the fruit. It is a simple description of the equation per se, and no attempt is made to explain the meaning:

N is the product of 3 and M. (Q1)
P is 5 more than Q. (Q2)

vii) Number Reversal

As in (vi) this is purely a description of the number relation but here the reversal error is also evident:

M is 3 times N. (Q1)
Q is 5 more than P. (Q2)

viii) Price or Weight

Although neither cost nor weight are mentioned in the problems, these descriptions include such extraneous references:

An orange weight is 3 times an apple's weight. (Q1)
The money of an orange is the money of an apple plus $5. (Q2)

ix) Manipulation of Equation

No attempt at a written sentence but the equation is either manipulated or some related mathematical expression is given:

$N/3 = M$; $3M + M = 4M$ (Q1)
$(Q+5) + Q = 2Q + 5Q$; $P - Q = 5$ (Q2)

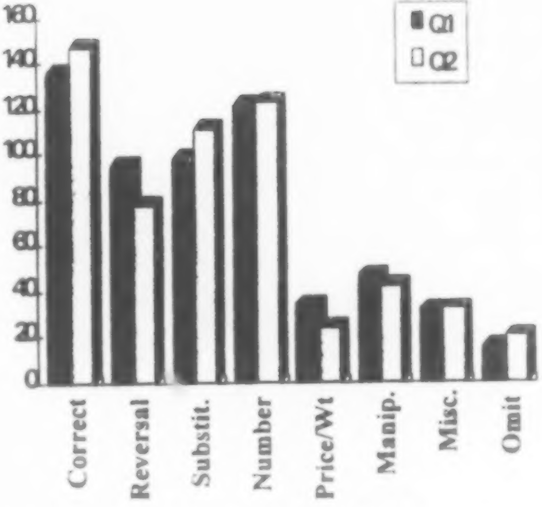
Results

It should be noted that in the above classifications some of the responses may be correct in a limited sense (e.g. "P is 5 more than Q" correctly describes "P = Q+5") but are not considered correct in terms of the requirements of the question. In the following table some of the classifications have been grouped together. Thus, "Correct" now incorporates categories (i) and (v), while "Reversal" incorporates categories (ii), (iii) and (vii).

Table 1 Distribution of Response Patterns

	Q1		Q2		Overlap
Category	Number	%	Number	%	(Q1)∩(Q2)
Correct	136	23.6	147	25.5	136(119)147
Reversal	95	16.5	78	13.5	95(54)78
Substitution	98	17.0	111	19.2	98(97)111
Number relation	122	23.6	123	21.3	122(115)123
Price/Weight	34	5.9	24	4.2	34(23)24
Manipulation	47	8.1	42	7.3	47(41)42
Miscellaneous	32	5.5	32	5.5	32(15)32
Omit	15	2.6	20	3.5	15(12)20

It can be seen from Table 1, and the bar-chart on the right, that the distribution of these responses is very similar for both problems. Moreover, the degree of 'stability' or 'consistency' across the two problems can be further illustrated by looking at the 'overlap' for each category. (This is shown in the last column of the table by the number in the brackets. That is, the number of students who answered both questions with the same type of response.)



Taking all the categories together, we find that 476 out of the 577 students (that is, 82% of the students) gave precisely the same type of response for both problems.

The facility levels for the 'construction' items on the tests ranged from a low of 50% to a high of 96%. (It will be recalled that the pupils in this study were 'average-to-high' in terms of academic achievement). Thus, the performance on these two items (with facility rates around 25%) represents a striking difference. Table 2 shows a comparison of these two 'interpretation' problems with the two lowest-facility 'construction' type problems, with the incidence of the reversal error highlighted. The latter problems are labelled Q3 and Q4 here for convenience:

Q3 In a college there are 10 times as many students as teachers. If there are N teachers and M students, write down an equation showing the relation between N and M.

Q4 In a classroom there are 6 more girls than boys. If there are N boys and M girls, write down an equation showing the relation between N and M.

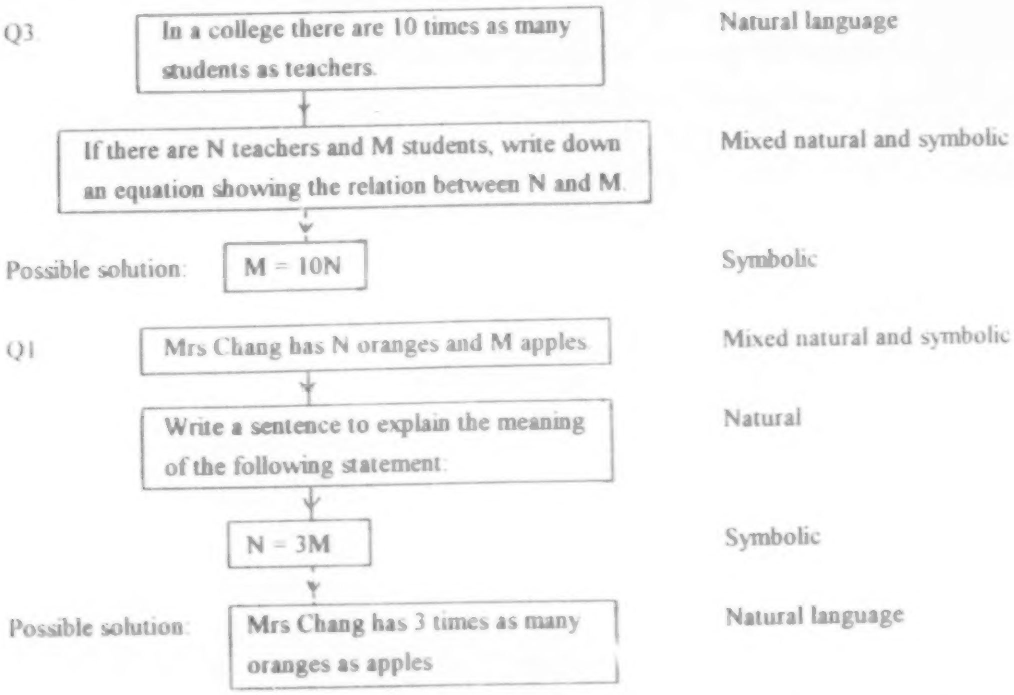
Table 2 A Comparison between Interpretation and Construction Items

	Correct	Total	Facility (%)	Errors	Reversal error	Reversal as % of Total	Reversal as % of Errors
Q1	136	577	24	441	95	17	22
Q2	147	577	26	430	78	14	18
Q3	97	194	50	97	63	32	65
Q4	116	194	60	78	44	23	56

(Note that the total number of responses for Q's 3 & 4 is not 577 since parallel versions of these problems were also given whereas Q's 1 & 2 were given to all pupils.)

Discussion

In the introduction it was anticipated that the interpretation of a mathematical statement into everyday language would be an easier task than the construction of a mathematical relation from a statement in natural language. For the cases examined in this paper, the results clearly show that interpretation can be at least as difficult, if not more so, than construction. Why should this be? First, the rather naïve analogy with decoding and encoding in natural language acquisition is patently not valid. In such a case, one is dealing with two processes acting on the *same* language whereas here we have a translation or transformation process between *two* languages, namely natural language and symbolic mathematical language. In fact, the situation is rather more complex than this description suggests since, in both the interpretation and construction problems, at least one of the sentences used involves both symbolic and natural language. Taking Q's 1 and 3 as examples we can diagramatically represent them as follows:



Q3.

In a college there are 10 times as many students as teachers. — Natural language

If there are N teachers and M students, write down an equation showing the relation between N and M. — Mixed natural and symbolic

Possible solution: $M = 10N$ — Symbolic

Q1

Mrs Chang has N oranges and M apples. — Mixed natural and symbolic

Write a sentence to explain the meaning of the following statement: — Natural

$N = 3M$ — Symbolic

Possible solution: Mrs Chang has 3 times as many oranges as apples — Natural language

There is little here to suggest an intrinsic reason why one translation direction should be more difficult than the other. However, we may note that the final required sentence can be considered more 'open' in the interpretation case than the construction. In Q3, once we have been instructed to write an equation relating N and M we have a clearly defined goal (whatever our capabilities may be in attaining it).

This was evident in the test papers where, apart from omissions and some idiosyncratic answers, over 90% of the students produced some form of equation or mathematical expression. However, it is clear from the response-patterns listed earlier, that in the interpretation problems no such clear goal was apparent to the pupils. It may also be a significant factor that Q's 1 and 2 begin with a mixed natural/symbolic sentence whereas Q's 3 and 4 set the scene with a purely natural-language sentence. This would need to be explored with further testing.

It would appear that the open-ended nature of the interpretation problems allows for a myriad other influences to act on students' solution attempts. And it can be argued that what is most revealed by these different response patterns are the underlying perceptions and assumptions that pupils hold with respect to mathematics and problem-solving. For example, consider the Number Relation category in which pupils completely ignored the reference to fruit. This is strongly suggestive of the divorce of mathematics from real-world contexts in the pupils' minds. If mathematics is really all about numbers and equations, then when asked to explain the meaning of a mathematical relation the obvious response is to give a literal description of the equation. Similarly, the Manipulation category suggests that for many pupils the over-riding perception of mathematics concerns performing algorithms. Unless one is actively operating on and manipulating numbers or symbols then one cannot be doing mathematics. The descriptions involving Price or Weight, which are brought in quite gratuitously, are indicative of pupils' attempts to link the questions to familiar 'text-book' problems. These pupils' strategies for problem-solving are almost certainly dependent on identifying key words and contexts (in this case, the association of fruit with shopping perhaps since, in sex-stereotype fashion, the protagonists are female). Finally, let us consider the Substitution category which uses both fruit and symbols e.g. "N oranges is the same as 3 times M apples" and "Q apples plus 5 is the same as P oranges". The reason for describing this as 'substitution' is that the attempt to explain the equation is met by simply replacing N and M by N oranges and M apples taken from the initial sentence. The two variables remain effectively unrelated in the final sentence since there is no appreciation that one is a function of, or dependent on, the other. This is particularly clear in the phrase "Q apples plus 5" where we are forced to ask "5 what?"

Table 2 superficially suggests that although the reversal error is still an important element in the interpretation problems, it is no longer as dominant as in the construction problems. Certainly this is so when we look at the reversal errors as a percentage of all errors. However, this may not be surprising in view of the fact that the facility rate is so much lower due to so many other error-responses coming into play with the interpretation problems. (It can be argued that in the construction case, *if* an error is made then it is likely to be the reversal error simply because the number of alternatives is limited). In any case, it is clear that the reversal error is still frequent enough to warrant serious attention, particularly in those cases where the students attempted to write a natural-language sentence.

Conclusions

A number of interesting questions and issues are raised by these results. First, although the students in this sample are competent in English, it must be remembered that it is their second language. The problems described as 'interpretation' here do indeed involve interpreting the algebraic equation but of course they also involve the *construction* of the natural-language explanation. Is this a significant difficulty for second-language students? Secondly, although the initial focus of this study was the reversal error, the results suggest that the interpretation of algebraic equations and expressions may be far more complex than supposed. It may well be that there is far too much emphasis in most mathematics curricula on the 'construction' aspects of mathematics (i.e. *doing* mathematics) and not enough on making our students mathematically literate (in the sense of being able to 'read' mathematical content meaningfully). In statistics there certainly has been a change of emphasis in recent years towards more interpretation and understanding of graphs and charts rather than their production.

In order to explore more fully pupils' abilities to interpret algebraic expressions and equations, further testing is currently being undertaken and it is hoped to report on this in the future.

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LETTING GO: AN APPROACH TO GEOMETRIC PROBLEM SOLVING

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This paper develops an argument concerning the use of dynamic geometry software in the teaching of geometry. The use of such software offers opportunities for creating the geometrical objects necessary for solutions to geometric construction problems and can give insight into relationships involved. A technique involving releasing constraints construction problem is described, which has curricular implications and raises issues of students' capabilities. The paper concludes with suggestions for further enquiry.

GEOMETRICAL CONSTRUCTION PROBLEMS

Geometric construction problems have a long history both in mathematics generally and in school curricula. They range from 'basic' constructions, "Construct a triangle given the lengths of each of the three side", to ones which involve the use of many supplementary geometrical objects and ideas. Traditionally, such construction problems have been solved by reasoning, and any new geometrical objects needed for the construction were called into being as a result of mental imagery, possibly augmented by drawing on paper. Where students were called upon to create constructions for themselves (rather than reproduce previously taught ones), the majority of students had difficulty in imagining which new objects would be helpful and so were unable to solve the problems without considerable assistance (see, for example, Schoenfeld 1987).

Polya (1962) produced an analysis of the principles underlying many such constructions, which he described as 'the pattern of two loci'. He illustrated this with the problem of constructing a triangle given its three sides. One side, *a*, is drawn with endpoints B and C. Then a circle of radius *b* is drawn with centre C and a circle of radius *c* is drawn with centre B. the intersection of the two circle – the two loci – gives the third vertex of the triangle.

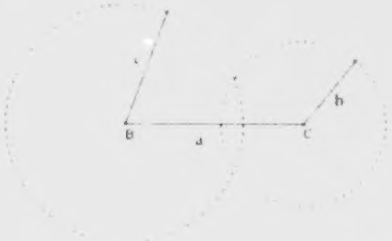

Fig 1 Constructing a triangle

Polya's analysis highlights two features. He claims that the first essential is to reduce the construction to that of creating a single point (the vertex A in this case).

> By laying down the segment *a*, we have already located two vertices of the required triangle, B and C; just one more vertex remains to be found. In fact, [...] we have transformed the problem to another problem equivalent to, but different from, the original problem. (Polya p.4)

The second feature is to '... split the condition into TWO parts, so that each part yields a locus for the unknown point; each locus must be either a straight line or a circle'.

However, the use of Polya's method requires that the loci are already envisaged, which, in turn requires a realisation of how they can be used in solving the problem. In effect, the problem is imagined as solved before the construction is embarked upon. It is this 'imagined as solved' feature which causes great difficulty for many learners.

LOCUS CONSTRUCTION IN DYNAMIC SOFTWARE

With the availability of dynamic construction software such as *Cabri-géomètre* and *Geometer's Sketchpad*, new techniques become available that do not rely on the learner visualising the loci in advance. Learners can create loci without previously having to imagine them, and these loci can be used as the basis for solving construction problems.

When using dynamic construction software, one can follow Polya by separating the conditions of the problem and consider them separately. In practice, this means dropping one of the conditions while constructing a figure subject to the other constraints. This creates a dynamic configuration – which is constrained by the other conditions. The key feature available in this software is the drag mode (see Laborde (1995)). When dragging a point, other geometrical objects move, retaining their relationships in the total figure; by then observing the loci of relevant points, the solution to the problem can be seen amongst the possible positions. Moreover, by obtaining and examining the loci, an actual construction method which can solve the problem can be inferred. To fix the discussion, the following problem will be used as an example.

> **Example 1**. Given a point A and two lines *b* and *d*. Construct a square ABCD with B on line *b* and D on line *d*.

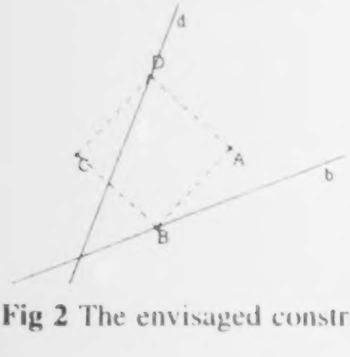

Fig 2 The envisaged constructed square

A method which uses the described features of the software is illustrated by the following solution:

> The condition that D should lie on line *d* is dropped, and a square is to be created with adjacent vertices at A and on line *b*. To do this, a variable point B is created on line *b* and a square ABCD is constructed on side AB.
>
> When B is dragged along line *b*, the locus of D appears to be a straight line, call it *m*. The intersection of line *m* with line *d* will give the desired position for D, and so give the square ABCD.

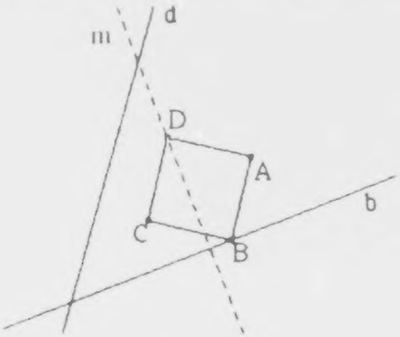

Fig 3 The locus of D

The locus of D has now been made overt: that it appears to be a straight line (and perpendicular to line *b*) now becomes part of appreciating the problem and a significant step in the solution.

Using the software to gain further insights

There is, of course, a further stage in solving the problem: the position of line *m* has to be determined. This requires finding a point on line *m*. The geometric software has a role to play in this also.

Dragging B to the foot of the perpendicular from A to line *b* (call this point P) gives rise to this configuration:

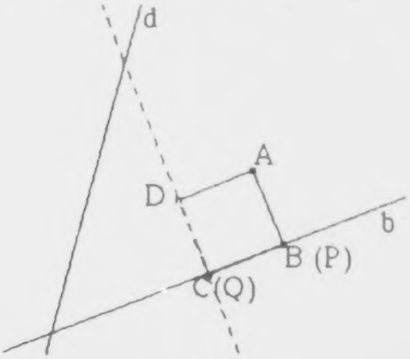

Fig 4 A particular position of B.

This indicates that line *m* cuts line *b* at point Q where PQ = AP. Again, the drawing suggests how the intersection of line *m* with line *b* is related to the distance of point A from line *b* and thus enables the intersection point to be constructed. Thus all of the elements are now available for the solution of the problem.[1]

Of course, while the software has assisted in solving the problem, it has not been proved that this is a solution. Such a proof requires insight into the relationships between the geometric objects. Further experimentation with the software can help here also. Insight can be gained by asking further questions and exploring in outcomes in the drag mode.

> What is the effect on line *m* of dragging each of the given elements in the diagram? That is, of moving point A; of moving line *b*; of moving line *d*. Moving point A moves the line *m* parallel to itself. Moving line *d* (whether rotating or translating) has no effect on line *m*. Moving the line *b* by translating moves line *m* parallel to itself; moving the line *b* by rotating rotates line *m*, apparently at the same rate, so line *m* remains perpendicular to line *b*.

The relationships evoked by these might lead to further insights. A solution to the original problem could then be stated concisely as:

> Construct P, the foot of the perpendicular from A to line *b*. Construct Q on line *b*, with PQ = AP. Draw the line *m* through Q perpendicular to line *b*. The intersection of this with line *d* gives the point D. Construct the square on the diagonal AD.

FEATURES OF THE METHOD

Thus, the assistance provided by the software is three-fold

- producing the geometric objects necessary for solution;
- relating those objects to the given conditions;
- giving insight into the relationships between the objects in the solution.

The essential feature of the method is to separate the conditions in the problem and drop one part of the condition. This enables the locus that plays a crucial role in the construction to be created.

What the approach offers

It might be argued that the solution above could have been obtained by insight and reasoning, and that no recourse was needed to dynamic geometry software. While this is so, the hitherto unknown elements required for solution are made visible by the software and so do not have to be simply imagined or deduced from other properties by the solver. This is invaluable for the student, who now has the elements of a solution at their disposal, rather than having to think what they might be. There is still the issue of justifying the results obtained experimentally, but unlike traditional approaches involving reasoning, this approach separates the use of reasoning to obtain the elements from its use in justifying the solution.

History of the approach

Although Polya was taken as the exemplar of this method, the ideas behind the method are very old. Polya claims that his method is foreshadowed in Descartes and, much earlier, Proclus. In recent times, the idea of letting go a constraint is closely connected to the method, developed by Brown and Walter (1983) of "What if not? Their technique consisted of listing attributes of a situation and asking what might be possible if one of the attributes did *not* hold. They used this technique as means of generating problems, whereas here the related version is used as a means of creating solutions. Goldenberg (1995) has advocated a method of examining geometrical theorems using dynamic construction software by 'seeing a theorem as a function or recasting static statements of fact to be dependencies on some variable element' (p.205).[2]

FEATURES OF THE APPROACH

Further implications of the 'letting go' approach are shown by two further examples.

(a) The method is not algorithmic.

> **Example 2**: Given: two concentric circles and a fixed point A, within the larger circle. Construct a line segment with one end on each circle and its midpoint at A.

Using a similar approach to this problem, it is possible to separate the conditions that the line segment has one end on each circle and also has its midpoint at A. Conditions can be dropped in one of two ways:

- drop the constraint that the segment must have its ends on the inner circle and find the locus of the unattached end.(Fig 4(a));
- drop the constraint that the midpoint must be at A and find the locus of the midpoint of the segment. (Fig 5(b)).

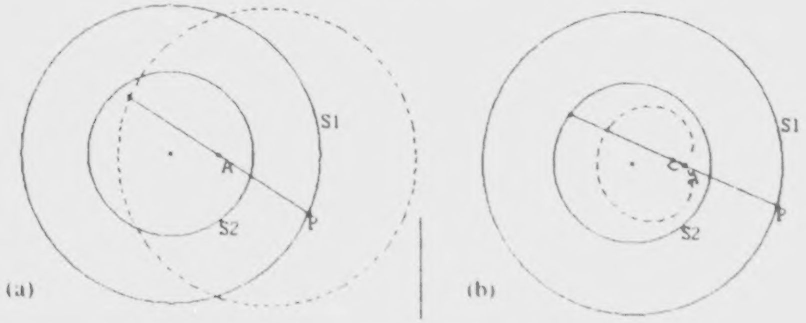

Fig 5 (a) and (b). Dropping different constraints

Here it matters crucially which constraint is dropped: dropping one constraint leads to a curve that is not constructible using traditional methods. Dropping a different constraint leads to a circle (in fact, of the same radius as the larger circle) and an easy construction. In the case of the construction implied by 5(a), experimenting with the given objects can again suggest further relationships.

(b) Objects used in the constructions.

In the traditional methods, ruler and compass alone are used. Where a problem cannot be solved using these methods, the software can still effect on construction.

> **Example 3**. Given two parallel lines, *l* and *m* and a point between them, A. Draw a circle that is tangent to both *l* and *m* and passes through A.

There are several ways of dropping a condition to solve this problem. One, perhaps not so obvious, is to create a variable point, M, on line *m* and create the locus of points which are equidistant from A and from M. Where this locus intersects with the line parallel to and mid-way between lines *l* and *m* will give the solution.

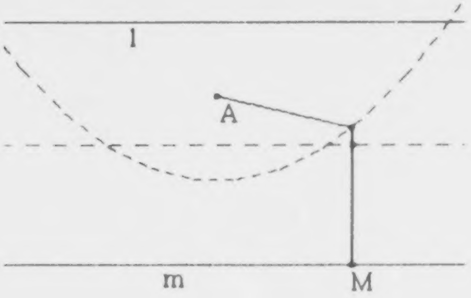

Fig. 6 A locus solution

The construction in this case leads to a parabola, which cannot, of course, be constructed with ruler and compasses. Polya dismisses a related solution to this problem because

> although splitting the condition into these parts is logically unobjectionable it is nevertheless useless: the corresponding loci are *parabolas*: we cannot draw them with ruler and compass – it is an essential part of the scheme that the loci obtained should be circular or rectilinear. (Polya (1962), p.6)

However, parabolas *can* be constructed in *Cabri-géomètre* and so the intersection point found. This construction raises the issue of what might be allowable objects in constructions.

SOME CURRICULAR IMPLICATIONS

The suggestibility of results

The figures obtained in the constructions are suggestive rather than definite. Indeed it is easy to think, mistakenly, that some locus is a particular kind of curve (a conic, say) when it is not. What the results do is to suggest certain relationships, which are then overtly available for reflection, discussion, amendment. Loci on the screen provide indicators: they are not definite means of construction.

Types of constructibility

In the examples above, some solutions to the problems relied upon ruler and compass methods, whereas others required the use of conic sections. Traditionally, the methods available for construction in school mathematics have been restricted to the use of ruler and compass. Largely because of the influence of Euclid on school mathematics, ruler and compass constructions came to be seen as the pre-eminent – and often the sole allowable – means of constructions. The earliest versions of the software being used here also restricted themselves to such constructions. The work of Knorr (1986) on problem-solving in ancient Greece has clarified – and overturned – this ancient legacy. Knorr demonstrates that, far from pursuing the use of ruler and compasses methods as a means to establish a formal programme of mathematical constructivism, the mathematicians of the time were engaged upon 'the activity of investigating problems of construction'.

> By the time of Apollonius, [...] the wealth of results permitted one to gain a sense of the structure of the field: which forms of conditions give rise to planar loci (circles and lines), which others to solid loci (conic sections), and accordingly, which kinds of problems are amenable to planar or solid solving methods. (Knorr p.369)

We can now assume that the restriction to ruler and compass was part of a project to classify problems involving constructions by the kinds of methods employed. The facilities available in *Cabri-géomètre* now allow constructions involving conic sections as well as circles and straight lines.[3] This offers a challenge to the development of school geometry: in what sense is a construction obtained by the intersection of a conic and a circle less valid, when the means of producing conics is available?

Psychological Issues

The inhibiting effects of students' perceptions of geometry on their geometric problem-solving have been outlined by Schoenfeld (1987). Although some of these difficulties may be mitigated when dynamic construction software is used, others are likely to remain, and it is possible that new ones will arise, as is suggested by Laborde (1995). For example, it may be conjectured that students are likely to have difficulty in separating the various conditions in the problem. With the locus-construction method outlined above, one additional difficulty is likely to be in ignoring one condition and still assuming that a solution can be found. On the one hand, it might felt that such a 'reduction of the cognitive load' would make a problem easier to deal with. However, this is to ignore the emotional cost of leaving out (leaving behind) some of the stipulations of the problem.

CONCLUDING REMARKS

To assess the viability of these methods by students a research programme is needed. Such research would need to focus on:

- how well students can separate the constraints in a construction problem;
- the psychological barriers to dropping constraints;
- what criteria they use for choosing the constraint to be dropped;
- whether, once the locus is found, together with sufficient means for constructing it, students appreciate the need for justification (and if not, how this need might be fostered);
- what are the problems for students associated with the methods of construction involving loci from ruler and compass (i.e. circles and straight lines) as against those involving conics.

It is hoped to examine these issues in a further paper.

NOTES

[1] One different method is to create points on the locus and use them to create the line *m*. The intersection point of line *m* with line *d* can then be found. This method raises other issues about the status of objects in *Cabri-géomètre*.

[2] The *Connected Geometry* project at the Educational Development Centre is attempting to recast the curriculum in such a way.

[3] Although *Cabri-géomètre* does not provide intersections between two conics, but only between a conic and a line or a circle.

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LEARNING TO FORMULATE EQUATIONS FOR PROBLEMS[1]

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We report an investigation of students' attempts to formulate equations for word problems. A sample of 90 students aged 14-16 was tested three times over a 10-month period. Some students used no algebra in any test, some tried to use algebraic formulation without success, and some were able to set up equations that could be used to solve the problems. In this paper we trace a progression from naming quantities through describing relationships to writing equations. Certain well-documented errors were not as common as expected. Integrating all the necessary information into one useful equation was a common difficulty. We discuss the effects of particular teaching approaches on the use of algebra for solving problems.

Over the past two years we have been investigating students' performance in formulating algebraic equations for word problems. In Stacey and MacGregor (1995) we reported our findings concerning the effects of problem presentation on students' solution strategies. The majority of students in our sample had not attempted to use algebraic solutions. However we observed that a considerable number of them used algebraic letters to name quantities in a problem or to label parts on a diagram, but did not go on to make use of the notation. To solve the problem they switched to arithmetic calculation or numerical trial and error. Others tried to express the relationships implied in the problem (i.e., the equivalence between the sum of three unequal parts and a given quantity) but did not produce a useful equation. One of the schools tested in 1994 allowed us to use the same test items on two more occasions with the same students. The test results indicate that, although students' use of algebra had progressed, the majority still did not formulate a correct equation and solve it. By comparing students' written responses over the 10 months between the first and last tests, we were able to see how students approached the task of formulating an equation for a problem, in what ways they improved, and what the difficulties were that prevented success.

Cortes (1995) identified several types of errors in the formulation of equations for word problems. Some of these errors were associated with (a) recognising relationships between quantities in the problem situation and writing them correctly, and (b) writing an equation or system of equations that would enable the unknowns to be calculated. One cause of error in representing relationships was the reversal error in writing an expression or equation for compared quantities (e.g., students wrote the incorrect $x + 120 = y$ to represent "x is 120 greater than y"). There is much evidence (Lochhead & Mestre, 1988; MacGregor & Stacey, 1993) that when students write algebraic equations and expressions about two compared quantities reversal errors are common. Cortes found that 30% of his student sample made the error in a problem-solving context. Another difficulty in representing information given in a word problem is the symbolisation of sums and products. Stacey and MacGregor (1994) found that in some situations students write conjoined terms for sums and use exponential notation for products. It is likely therefore that representing the separate relationships given in a word-problem, - an early step towards formulating the final equation or system of equations - is a major obstacle for students. Integrating these relationships as an equation in one variable is another crucial step, as Cortes has found. In the present study we looked at students' progress in these two steps in writing equations to solve word problems.

Test items

The tests comprised six algebra word problems varying in difficulty. Students were asked to write an equation for each problem and solve it. In Figure 1 we show three of the problems, and we refer to them in our discussion of students' difficulties.

Bednarz and Dufour-Janvier (1994) used problems of this type ("unequal partition" problems) for testing students with different amounts of algebra experience. They found that when a problem was worded so that two quantities could be expressed directly in terms of the first one, students found it easy to solve; when this was not the case, problems were harder to solve. According to Bednarz et al.'s data, Problem 1 (see below) would be far more difficult if it were expressed so that Monday's distance was related to Sunday's distance instead of to Saturday's. In the easy version, both Monday's journey and Sunday's journey are related directly to Saturday's journey. We have used the easy problem format for all three of our problems, thus ensuring that difficulties in comprehension cannot be attributed to avoidable linguistic factors.

WORK OUT THE ANSWERS BY WRITING EQUATIONS AND SOLVING THEM

1. A group of scouts did a 3-day walk on a long weekend. On Sunday they walked 7 km farther than they had walked on Saturday. On Monday they walked 13 km farther than they had walked on Saturday. The total journey was 80 km. How far did they walk on Saturday?

2. Jeff washes three cars. The second car takes 7 minutes longer than the first one. The third car takes 11 minutes longer than the first one. Jeff works for 87 minutes altogether. How many minutes does he take to wash the first car?

3. The three sides of a triangle are different lengths. The second side is 3 cm longer than the first side, and the third side is twice as long as the first side. The side lengths add up to 63 cm altogether. How long is the first side?

Figure 1. Test items

Subjects and Results

The students were in Years 9 and 10 (age 14-15) for the initial test, and Years 10 and 11 (age 15-16) for the final test. All classes involved were mixed-ability classes. Test papers showing all written working and answers were obtained from 90 students who had done both tests. Their progress, judged on all six problems, is shown in Table 1. Students' best performance is represented. For example, if a student used algebraic letters to record given information in only one item of a test and used no algebra at all in the other items, this student is counted in the "Partial Use" category for that test.

In Table 1, "No algebra" means that there was no attempt to use any algebraic notation. "Partial Use" means that the student had attempted to use some algebraic notation, even if only to denote an unknown quantity by a letter (e.g., labelling a diagram for Problem 3 with the letter x on one side). "Equation" means that a correct equation was written. By "correct equation" we mean an equation in one variable that could have been used, or was used, to solve the problem (e.g., $x + x + 3 + x + x = 63$ for Problem 3).

Table 1

Progress in students' use of algebra over 10 months ($N = 90$)

	Final test		
Initial test	No algebra	Partial use	Equation
No algebra	53	11	2
Partial use	2	9	8
Equation	0	0	5
TOTALS	55	20	15

For each problem approximately 70% of students wrote correct numerical answers. Most numerical answers to all the problems, whether right or wrong, were obtained by non-algebraic methods, as used by the younger students in Bednarz et al.'s (1994) study. As Table 1 shows, there were 35 students (39% of the sample) in the final test who attempted some algebra. Five students formed and solved equations correctly in all three tests. However three of the five did not use standard algebraic techniques for solving their equations. For example, one used a systematic trial-and-error routine, trying different values for x in his equation.

It might be argued that since algebraic methods are not necessary for solving the problems, students who were capable of using algebra may not have done so and therefore have been wrongly classified in the "No algebra" category. However algebra was required for the harder problems in the test. The students who demonstrated their capability to use algebra for these harder problems had used it for all the problems.

We had expected that in the time between the tests many students would move from using no algebra to being able to formulate equations. However, as Table 1 shows (see first row), 53 students continued to make no attempt to use algebra, 11 moved from "No algebra" to "Partial use" and only two students moved from "No algebra" to writing a correct equation. The second row of the table shows that eight students moved from being partial users to writing an equation, and nine did not progress. There were 20 students at the time of the final test who used some algebraic notation but had not yet progressed to a stage where they could write an equation for these relatively straightforward problems. Two students used some algebra in early tests but no algebra in the final test. Their apparent regression can be attributed to the effects of teaching. Their responses, and the responses of other students in their class, indicated that they had been trained to use a systematic and successful trial-and-error approach to problem-solving. This training may be a reason why so few of the partial users of algebra did not progress, and why so many did not use algebra at all in either test.

Students attempted to deal with the information given in the problem in many different ways. Some of these were helpful ways of recording, whereas others contained specific well-documented errors that obstructed progress towards writing an equation. Stages in this progression, and examples of difficulties at each stage, are shown below. All the examples are chosen from students' written responses to Problem 1, unless stated otherwise.

1. Naming the unknown quantities referred to in the problem

The use of letters as abbreviated words, widely referred to in the literature, was a cause of difficulty for only three students. For example, to begin Problem 3 one student wrote:

$$\begin{aligned} 3 \text{ car} &= 1c + 11 \\ 2c &= 1c + 7 \\ 3c &= 87 \end{aligned}$$

He has written "3 car" to mean "time taken for the *third car*", "$1c$" to mean "time taken for the *first* car", and $3c$ to mean "total time for *three cars*". He understands the relationships in the problem situation, but has not understood the crucial difference between letters as abbreviated words and letters as representing quantities or variables. It is widely recognised that in certain circumstances students tend to use algebraic letters as shorthand names for objects. However for the problems used in this study, only three students used letters as names, as illustrated above. For the majority, identifying and naming unknowns was not a significant obstacle.

2. Expressing the relationships between the parts

Many students were able to express the relationships in the problem situations, often in correct algebraic notation (e.g., x, $x + 7$, $x + 13$) . For Problem 3, diagrams were frequently drawn and labelled appropriately. However there were some instances of well-documented errors in notation that blocked further use of an algebraic method. These errors were (a) reversal, (b) concatenation for addition, and (c) exponential notation for a product.

The reversal error was seen in the work of only two students; for example, $a, b + 7, c + 13$ to indicate that a is the smallest quantity, b is larger by 7, and c is larger by 13, instead of the correct a, b, c, where $b = a + 7$ and $c = a + 13$. Concatenation for addition was seen in the work of four students; for example, $x7$ and $x13$ to mean "7 more than x" and "13 more than x". Exponential notation for a product was seen in the work of six students; for example, x^2 to mean "twice x".

All these errors blocked further development of an algebraic procedure. Summarising the data above, we see that of the 37 students who attempted to use some algebra, 10 students (approximately one-quarter of the sample) were prevented from writing correct equations by their misunderstandings of algebraic notation for sums and products. Their misuse of notation may indicate a poorly-developed concept of multiplication and its relationships with repeated addition and repeated multiplication (Stacey & MacGregor, 1994). The reversal error was far less frequent than expected, given Cortes (1995) finding of a 30% rate of reversal in formulating equations for word problems.

3. Writing a useful equation that integrates the problem information

Some students wrote correct equations or expressions for all four relationships but did not combine them into an equation in one variable. For example, for Problem 3 a student labelled the three sides of her diagram of a triangle as

A
$B = A + 3$
$C = 2 \times A$

and then wrote the equation $A + B + C = 63$. She did not make any use of this equation and did not solve the problem. Another student wrote, for the same problem,

$S1 = x, \; S2 = x + 3, \; S3 = 2x$

Although this also looks a useful beginning, where S1 means "Side 1" and correct expressions in terms of x have been written for each side of the triangle, the student wrote as his equation $S1 + S2 + S3 = 63$. He then abandoned any further use of algebra and solved the problem by a trial-and-error method.

Several students did not know how to write equations in the standard way, although they appeared to understand what relationships were involved and how to integrate them. They were able to solve the problems by non-algebraic methods, and perhaps had used their "equations" in some way to guide reasoning and calculation. Examples are shown in Figure 2. These eight examples represent almost one-quarter (8/37) of the students who tried some algebra. These students knew that they should represent the sum of three parts, expressed in terms of one variable. It seems likely that with appropriate instruction they would quickly learn how to write equations in the standard way. It is puzzling why they had not learned to take this small last step.

(i) Sat Sun Mon

$x \to x + 7 \to x + 13 = 80$

(ii)

$$\begin{array}{l} x \\ x + 7 \\ \underline{x -13} \\ \;80 \end{array}$$

(iii) $\boxed{} \to \overset{+7}{\boxed{?}} \to \overset{+13}{\boxed{?}} \to \boxed{80}$

(iv)
$1 = y$
$2 = y + 7$
$3 = y + 13$
$1 + 2 + 3 = 80$

(v) $x,\ x + 7,\ x + 13,\ 80$

(vi) $\xleftrightarrow{\quad 80 \quad}$

$a = ?$ $b = a + 7$ $c = a + 13$

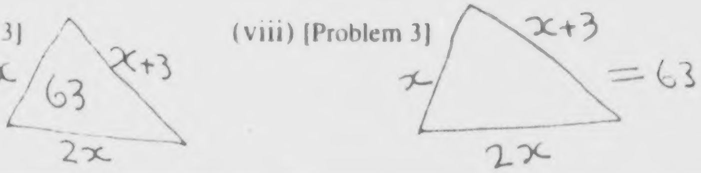
(vii) [Problem 3] (viii) [Problem 3]

Figure 2. Attempts to integrate problem information

4. Equations as descriptions of procedures used for calculating

Several students calculated answers to each problem by arithmetic reasoning, and then wrote their calculations as "equations". This method of dealing with algebra word problems has been observed by other researchers (Arzarello, Bazzini & Chiappini, 1993). The pseudo-equations were not representations of problem structure, but descriptions of the procedures students had already used to get each answer. For example, the equation for Problem 3 was written as

$x = (63 - 3) + 4.$

It can be argued that this is an acceptable equation for that problem. However in the harder problems on the test, which were too difficult to solve by mental reasoning and arithmetic, students who were limited to writing a description of the solution method had no chance of success. Their reluctance or inability to engage in algebraic thinking restricted them to carrying out a sequence of arithmetic procedures with a numerical answer at each step.

Discussion

In much of the literature, it is suggested that failure to solve a word-problem is often caused by not comprehending the problem situation. Our data indicate that major difficulties in formulating equations in our test did not lie in students' failure to comprehend the written information, to understand the problem structure, or to see how the parts were related to each other and to the whole. Most students could solve the problems by non-algebraic methods, providing confirmation that understanding the problem situation was not a difficulty. Their difficulties were due to misuse of algebraic notation, including not knowing how to write an equation.

As Arzarello et al. (1993) have commented, writing equations for problems is a complex process. These researchers have suggested that the process of naming (i.e., the choice of variables) and understanding the main relations in the problem are the most crucial steps. Our data indicate that naming variables and understanding relations were not difficult for the simple problems we used. Knowledge of how to use algebraic notation to express this understanding was a far more important factor in the equation-writing process. Most students who tried algebra could name quantities, and there was little difficulty related to expressing several quantities in terms of one variable. When writing relationships, about one-third of students experienced difficulty, although the well-documented errors of reversal and concatenation were not the major causes. As we have shown, students used unconventional formats such as arrow-diagrams, vertical addition, or invented notations to try to denote the equivalence between three parts and their sum. They were unable to write an equation to integrate relationships they had already deduced correctly from the information given. Others had not learned that an equation is written to represent the problem situation; they wrote a description of the calculation procedure they had used.

In a typical school algebra curriculum, the first problems given to students to solve by algebraic means can also be solved by simple arithmetic. Until they achieve a certain level of fluency, students see algebra as an extra difficulty or unnecessary task imposed by teachers for no obvious purpose and not as a useful tool for making problem-solving simpler. This attitude is reasonable, since the problems they have so far encountered (such as the three problems presented in this paper) are not good examples of the power of algebra. Cortes, Vergnaud & Kavafian (1990) state that if students are to learn to formulate algebraic equations for solving problems, teachers need to discourage the search for arithmetic solutions. We support this view, while reminding readers that it is difficult to find problems that are sufficiently complex to warrant an algebraic solution but easy enough for students to work through with understanding and learn from. We suggest that simple problem situations with harder numbers would help students to see the advantage of an algebraic method (e.g., changing the perimeter of the triangle in Problem 3 from 63 cm to 61.8 cm makes it more difficult to solve without algebra).

In Australian schools some students get very little experience in solving problems by formulating equations. Methods preferred by many teachers are the trial-and-error approach, which bypasses the need to learn algebraic techniques and even avoids the need for writing an integrated equation to represent a problem. The use of spreadsheets and graphical methods, also currently advocated, is intended to develop ideas of equations and functions, of substituting values in expressions, and of understanding what a solution to an equation or set of equations really is.

Routine manipulation methods for solving equations are not needed for solving problems by these methods, but even so the equations still have to be formulated. A question for teachers is how to give the majority of their students a sound understanding of algebraic ideas firmly grounded in numerical instances, whilst at the same time helping them to overcome their reliance on arithmetic processes and develop expertise in the use of formal algebra.

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ORIGINS OF STUDENTS' INTERPRETATIONS OF ALGEBRAIC NOTATION

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Research studies have extensively documented students' misinterpretations of algebraic letters. The principal explanation given in the literature has been a general link to levels of cognitive development. In this paper we provide more specific explanations for particular misinterpretations by analysing written and oral responses from secondary school students to four simple algebra items. The 24 schools involved had followed different teaching programs. Students' ages ranged from 11 to 15. We trace the origins of misinterpretations to the making of intuitive assumptions and pragmatic reasoning about the unfamiliar, to drawing analogies with familiar symbol systems, to interference from new learning in mathematics, and to the effects of misleading teaching materials.

The research literature on algebra learning extensively documents students' difficulties in learning fundamental aspects of algebraic notation such as how to write simple expressions and equations containing letters, numerals, operation signs and brackets (e.g., Assessment of Performance Unit, 1985; Booth, 1984; Cambridge Institute of Education, 1985; Herscovics, 1989; Küchemann, 1981). Küchemann classified students' interpretations of algebraic letters into two major divisions:(i) letter ignored, given an arbitrary value, or used as the name of an object; (ii) letter used as a specific unknown number or as a generalised number. Küchemann suggested that these interpretations were associated with Piagetian stages of cognitive development. Nevertheless there is evidence that cognitive level is not a sufficient predictor of success in interpreting algebraic letters. Firstly, in the *Concepts in Secondary Mathematics and Science* [CSMS] research project (Hart, 1981), some students with below average IQ scores reached unexpectedly high levels of understanding in algebra. Secondly, many mathematics educators at the present time recommend an approach to algebra that depends on students' ability to grasp the concepts of generalised number and unclosed expression. Thirdly, the success of students in experimental computer environments (Cohors-Fresenborg, 1993; Sutherland, 1991; Tall & Thomas, 1991) suggests that at least some of the difficulties and errors in traditional algebra learning are caused by the nature of students' learning experiences and do not reflect their cognitive capacities. These counter-indicators suggest that other factors besides cognitive level need to be taken into account when explaining students' understanding of algebra.

In the many years since the CSMS project, it has been widely accepted that cognitive level is a sufficient explanation for the ways in which algebraic notation is interpreted. If cognitive level is viewed as a barrier preventing the construction of certain concepts, it explains why students cannot do certain algebraic tasks. However it does not explain why they interpret the notation in particular ways and why they make certain errors. In this paper, we outline some of the reasons behind specific misinterpretations, exposing factors which are more accessible to improvement by teaching than is cognitive level. We present evidence that the origins of students' interpretations include:

- intuitive assumptions and sensible, pragmatic reasoning about an unfamiliar notation system;
- analogies with symbol systems used in everyday life, in other parts of mathematics or in other school subjects;
- interference from new learning in mathematics;
- poorly-designed and misleading teaching materials.

Research questions and testing

In this short paper we analyse student's responses to a small number of items (see Fig. 1) which are concerned with the interpretation of algebraic letters and the writing of simple unclosed expressions.

DAVID

David is 10 cm taller than Con. Con is h cm tall. What can you write for David's height? ....

SUE

Sue weighs 1 kg less than Chris. Chris weighs y kg. What can you write for Sue's weight? ....

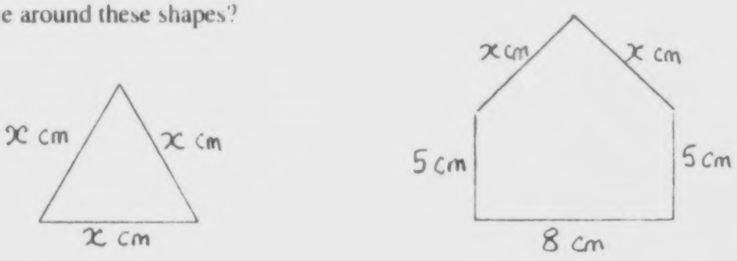
TWO OPS

n stands for an unknown number. Write the following in mathematical symbols:

"Add 5 to n, then multiply by 3" ....

DISTANCE

What is the distance around these shapes?



(i) ....   (ii) ....

Figure 1. The four items discussed in this paper

We discuss the following questions:

1. How do students who have not learned any algebra interpret letters and try to write expressions?
2. How do students' interpretations of letters and simple algebraic expressions change over three years of school algebra learning?
3. What are the roots of specific errors and misunderstandings?

The data in this paper are drawn from pencil-and-paper tests given to a large representative sample of approximately 2000 students in years 7-10 (ages 11-15) in

24 Australian secondary schools. Some schools used the test across year levels, thus providing comparative data. Other schools tested the same cohort of pupils on two or three occasions, providing us with longitudinal data on individual students. Parallel versions of the items (not shown here) were used when the same individuals were re-tested. At one school 14 students were interviewed and audio-taped while working on selected items. The schools involved were not randomly selected, but because of the sample size, the number of schools, and the range of school types (State, Catholic and private, in working-class and middle-class suburbs), there can be little doubt that those findings which are common to all the schools apply to the general population of students. Results which are not uniform across schools point to the influence of factors specific to particular schools.

In the following discussion of results, we first look at the way in which algebraic letters were interpreted by students who had not been taught any algebra, and discover that they tended to make intuitive assumptions or draw on analogies with familiar symbol systems. These interpretations were subsequently found to be made by more experienced students. Next we report the results of tests used for several hundreds of students in Years 7 to 10 in 22 schools, all of whom had studied some algebra. Some of the misinterpretations and errors made by older students were not observed in younger students, indicating interference from new learning. Finally we trace the progress of 156 individual students in three schools who were tested three times, and link their progress to features of the teaching programs.

Intuitive assumptions and analogies with other symbol systems

To examine students' unschooled ideas about algebraic letters, the algebra items were included in a test for two mixed-ability classes ($n = 42$) of Year 7 students (age approx. 11-12 years) who had not been taught any algebra at school. We expected that most if not all these students would not attempt the items containing algebraic letters. If answers were written, we expected them to be at Küchemann's (1981) lowest level (i.e., letter ignored, given a numerical value, or used as a label for an object). Two-thirds of the students did not write any answers, but the responses of the other 14 are useful indicators of students' intuitive interpretations of what algebraic letters might mean. Table 1 shows the responses to the item DAVID and the likely explanations for them.

We see in Table 1 nine sensible answers to what must have seemed a strange question. Two students used h to represent a quantity to which 10 cm could be added, and one wrote the correct expression $10+h$. One student used letters as abbreviated words. Three students attended to the alphabetical position of h, two of them deriving a numerical value as they often have to do in puzzles and codes and as was used in Greek numeration ($\alpha = 1$, $\beta = 2$, etc.). Two students reasoned that if Con's height could be represented by a letter, then so could David's. Two students assumed they should take a value for Con's height, since it was not given. The other five students, not knowing what to do, had tried to write something related to the numbers in the question, ignoring the letter. Table 2 shows the responses of these

14 individuals to the two items, and explanations for their responses to SUE. It is clear that students' reasoning across the two items was highly consistent.

Table 1. *Responses to item DAVID from 14 students*

Frequency	Response	Assumed reasoning
1	$10+h$ [correct]	Add 10 to number or quantity denoted by h.
1	$h10$	Add 10 onto h.
1	Uh	Abbreviated words "Unknown height".
2	18	h is the 8th letter of alphabet, therefore 10 more is the 18th.
2	t, g	Choose another letter or adjacent letter for David's height.
2	110	Think of a reasonable height for Con, add 10.
5	10,20,"half"	No comprehension of the question; use of the given value 10 and operations "double" or "half".

Table 2. *Responses to items DAVID and SUE from 14 students*

Frequency	DAVID	SUE	Assumed reasoning for SUE
1	$10+h$	$y-1$	Subtract 1 from number or quantity denoted by y.
1	$h10$	x	Although 10 can be "joined" h, as $10h$, 1 cannot be "removed" from y. To denote 1 less than y, write x.
1	Uh	Uw	Abbreviation for "Unknown weight".
2	18	24	y is the 25th letter and 1 less is 24.
2	110	[no response]	-
2	t, g	o, x	Choose another letter or adjacent letter.
4	10, 20	1	No comprehension of the question; use of the given value 1.
1	"half"	[no response]	-

These students displayed responses from Küchemann's lower (*letter ignored, abbreviated word,* and *numerical value assigned*) and higher (*letter as unknown quantity*) levels and drew upon analogies with familiar symbol systems and codes.

Interference from new learning

In this section, we demonstrate that older and more experienced students often misinterpreted and misused algebraic letters as a result of interference from new learning in mathematics. The four items were included in tests used by 22 schools for Years 7 to 10. All students in this sample had been taught some algebra. The Year 10 students were more successful than the Year 7 students, but there was not the great improvement that we had hoped for. Table 3 gives the success rates.

Table 3. *Percentages of students in Years 7-10 correct* ($N = 1463$)

Item [answer]	Yr 7 (n=307)	Yr 8 (n=511)	Yr 9 (n=338)	Yr 10 (n=307)
1. DAVID [$h+10$]	39%	52%	63%	73%
2. SUE [$y-1$]	36%	46%	60%	64%
3(i) DISTANCE [$3x$]	42%	44%	65%	61%
(ii) DISTANCE [$2x+18$]	27%	35%	55%	53%
4. TWO OPS [$3(n+5)$]	14%	17%	25%	47%

At all year levels, letters were used with several different meanings, and included the intuitive interpretations made by Year 7 beginners. However, new misinterpretations appeared. They were:

- as a ***label*** associated with an object or quantity (e.g., C to mean "Con's height" and D to mean "David's height" in $C+10 = D$).
- *letter equals 1* unless otherwise specified (e.g., $10+h = 11$).
- letter has ***a general referent*** that includes various specifics (e.g., h means "height", so it means both "David's height" and "Con's height" in the statement $h = h+10$).

Furthermore, a few students seemed to believe that if a coefficient is on the left of the letter it indicates subtraction and if it is on the right it indicates addition. They wrote $h10$ to mean "add 10 to h" and $1y$ to mean "take 1 from y". This notion may come from their knowledge of Roman numerals (as in VI for 6, and IV for 4), or from their experience with adding and subtracting along the number line (to add, move right; to subtract, move left), or may be based on intuitive metaphorical concepts associated with addition and subtraction (Lakoff & Johnson, 1980).

Older students had more opportunities for making mistakes than younger ones because of interference from new schemas only partly learned or because of their expectations of being able to use more advanced knowledge. For example, the misuse of exponential notation (x^3 instead of $3x$) increased steadily over the four year levels, from 5% on one item at Year 7 to 18% at Year 10. When students were interviewed on DISTANCE, they made comments such as "That's the hypotenuse", or "If it's 8 across that way, then rule off the line and cut straight up". It was interesting to see that for part (ii) several Year 10 students wrote x^2+5^2+8. Their uncertainty about how to write "twice x" may have contaminated their knowledge of how to write "twice five", and their expression containing a sum of squares superficially looks like Pythagoras's theorem.

Younger students often ignored the algebraic letter and chose a number for Con's height in DAVID. For DISTANCE some measured the lengths marked "x cm" with their rulers. They probably thought this was what the teacher wanted. Our data indicate that some numerical responses from older students were not due to the use of arbitrary numerals or measuring. An example is the following, written by a student in Year 10 and producing the answer 5 for DAVID:

$$10 + h \qquad \frac{10 + h}{h} \times \frac{h}{h}$$

$$= \frac{2h}{2} + \frac{10}{2}$$

$$h = 5$$

This student has written the correct expression $10+h$ but has then tried to use routine manipulation techniques, possibly recently learned. It is possible that numerical responses of this type have been misclassified in previous studies as *letter ignored* or *arbitrary numeral*.

Another source of numerical responses, also undetected in previous studies, is the belief developed by some students that any letter stands for 1. This was strongly evident in two schools in our sample but rare in others, indicating that it is probably partly due to aspects of instruction. The origins of this belief became clear in the interviews. A Year-10 student said, "x is just like 1, like having one number". Another said "By itself it is 1, the x". For DISTANCE (ii) a student worked out the answer as 20, and explained that 8 plus two 5's is 18, then "1 more for each x makes 20". One likely cause of this belief is a misunderstanding of what teachers mean when they say "x without a coefficient means $1x$". The student gets a vague message that the letter x by itself is something to do with 1. Another cause of misunderstanding is the fact that the power of x is 1 if no index is written (i.e., $x = x^1$). Answers that we had first classified as arbitrary numerical value (e.g., David's height = 11) or inaccurate measurement (e.g., 20 cm for DISTANCE (ii)), could in many cases be attributed to the *letter equals 1* belief.

Misleading teaching materials

At three schools, teachers tested their students three times, twice in one year and once the following year. Table 4 shows results for the 156 students who did all three tests on the three items DAVID, SUE and TWO OPS. As noted before, superficial differences were made to the items in the later tests.

Table 4. *Percentages correct in groups each tested three times* (n = 156)

	School A, Yr.8-9 (n=70)			School B, Yr.8-9 (n=60)			School C, Yr.9-10 (n=26)		
Item	1st	2nd	3rd	1st	2nd	3rd	1st	2nd	3rd
DAVID	70	86	96	70	90	90	50	69	65
SUE	57	90	93	75	95	90	38	65	46
TWO OPS	20	41	74	35	35	55	31	81	62

No teaching strategies or learning materials were suggested to the teachers at the three schools, except as discussed below. Trends in success rates on the table entries are therefore mostly due to the normal teaching that took place. As Table 4 shows, students at Schools A and B made good progress in coping with algebraic letters and unclosed expressions (DAVID and SUE), whereas School C continued to have difficulty. Success on TWO OPS improved at all schools, with a dramatic rise in the performance at School C. The reasons for these trends are explained below.

Teaching strategies and misinterpretation of algebraic letters

Misinterpretation of algebraic letters was a persistent difficulty at School C over all year levels. At Year 9, several students were guided by alphabetical order and wrote R for DAVID (i.e., ten letters after H) and X for SUE (i.e., one letter before Y). This error had not been recognised by the teachers concerned, and was easily corrected after it was brought to their notice. Other misinterpretations, in particular the use of letters as abbreviated words (so that "Con's height" was represented by C or C*h* but not by *h*), were more resilient. Discussion with the teachers revealed that teaching materials that had been used in Years 8 and 9 for these students explicitly present letters as abbreviated words (e.g., *c* could stand for "cat", so 5*c* could mean "five cats"). In contrast, teaching materials used at Schools A and B consistently present letters as standing for unknown numbers. In the data from these two schools, there were only two instances of letters used as abbreviated words in the first test and none thereafter. It seems probable that the widespread and persistent misinterpretation of letters as abbreviated words by the School C students can be attributed to the misguided teaching approach that had been used.

Teaching students to co-ordinate two operations

The item TWO OPS initially had a low success rate in the three schools and there was a very great range of incorrect responses. We had expected that omission of brackets (giving $n + 5 \times 3$) and the use of conjoining for addition (giving $5n \times 3$, $15n$, or $5n^3$, for example) would account for most errors, but this was not the case. We have no explanations for students' reasoning behind many of the other forms of incorrect response, and the large percentage of omissions at all levels.

At School C, between the first and second tests, teachers used a lesson designed by the researchers to address difficulties in coordinating two operations as required in the item TWO OPS. This lesson used examples of English text as well as mathematics to make students aware of the potential for ambiguity in certain expressions. The lesson stressed that mathematical statements cannot rely on the support of context which reduces ambiguity in natural language. For example, in the phrase "French men and women" the word "French" modifies both "men" and "women". However in the phrase "French fries and coke" the word "French" is not a modifier of the word "coke". In the phrase "Twice five plus three" it is not clear whether "twice" modifies 5 only, or both 5 and 3. Students were given practice at generating expressions of this type, inserting brackets to resolve ambiguity, and evaluating them. The teachers used this lesson to teach the use of brackets for grouping and distributivity, and they were clearly very effective (see Table 4). In the third test some months later, several students omitted brackets from otherwise correct expressions, suggesting that their knowledge of the purpose of brackets had not been used and was consequently forgotten. There were however no other types of error, in contrast with the great variety of errors seen elsewhere. We conclude that the lesson had been effective for the majority of students. We are not sure why it also seems to have been effective in eliminating the other types of error. The reason may be that students had learned to focus more clearly on what an algebraic expression means and to see how a slight change in notation affects this meaning.

Conclusion

We have shown that some common misunderstandings are the results of particular teaching approaches, and can be avoided. Others have been developed by the students themselves, with origins in the drawing of analogies, use of pragmatic reasoning or interference from new learning. Whilst some algebra errors are notoriously resilient to change, our research has identified some errors which are quite easy to fix. If teachers are made aware of the beliefs and assumptions about letters and mathematical notation that students bring with them to algebra learning, they can take account of these sources of misinterpretation in their teaching.

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MATHEMATICAL BELIEFS BEHIND SCHOOL PERFORMANCES

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A study on mathematical beliefs examined the important components of seventh-graders' belief structures and their effect on the mathematical performances. The data based on a self-report instrument and student scores in four mathematics tests included 476 seventh graders from 25 classes and 19 school over Finland. Factor analysis of the belief data revealed three main factors indicading self-confidence in mathematics, efficiency in mathematics learning and external view of mathematics, and of these high self-confidence was clearly the best predictor of performances.

The role of mathematical beliefs in affecting mathematical problem solving performances has been widely recognized. According to the given classifications and definitions (e.g. Garofalo, 1989; McLeod, 1989; Schoenfeld,1985) these beliefs can be divided into three main categories: beliefs about the nature of mathematics and mathematical tasks, beliefs about mathematics learning and teaching, and beliefs about oneself as mathematics learner and knower. Each of these sets carry important culturally and socially determined values and conceptions of mathematics which are developed and reflected in mathematics learning situations. The significance of mathematical beliefs and belief systems can be attached to the self-regulatory or metacognitive aspects of mathematics learning (e.g. Garofalo & Lester, 1985; Schoenfeld, 1987). Thus firmly established mathematical beliefs function as directive constructions which importantly affect the use and further construction of mathematical knowledge and skills. Moreover, mathematical beliefs seem to create a framework for students' powerful affective responses toward mathematics, as well as for such self-regulatory behaviours as decisions to persist in mathematical performances or choice of mathematics courses (e.g. Fennema, 1989; McLeod, 1989).

Beliefs included in the three categories interact with each other. Thus for example, students' beliefs about the objects or practices of mathematics learning can be derived from their beliefs about the nature of mathematics and mathematical problems. And again beliefs about oneself in mathematics are interwined i.a. with beliefs about mathematical problem solving or mathematical ability (Garofalo & Lester, 1985; Lester et al., 1989; McLeod 1989). When we come to the motivational aspects of mathematics learning, beliefs about oneself become important. "Beliefs about oneself and motivation are inseparable." (Underhill, 1988). Constructions of self in relation to mathematics form the basis especially for the important self-regulative (metacognitive) learning actions as persistence in mathematics or choice of mathematics (Fennema, 1989; McLeod, 1989). As another influential motivational part of mathematical belief systems has been suggested the so-called external vs. internal orientation in mathematics. Externally oriented students tend to view mathematical ability as fixed entity and mathematics performances as indicators of one´s intelligence, and thus prefer ego enhancement in their learning goals. Internal orientation instead is connected to the incremental view of mathematical ability, internal rewards as mastery of mathematical tasks, and often to high effort expense. (Dweck, 1986; Kloosterman, 1988) These two different orientational basis together with students´ self-confidence in mathematics and few other belief constructions has been studied against students´ mathematical achievements below.

Method

Subjects and Execution

The sample included 453 seventh-grade students (age 13) from 19 ordinary lower secondary school and 25 classes over the country with 219 girls and 234 boys, and with varying levels of mathematical achievements. The data was collected in connection with an international reseach project named the so-called Kassel project (Blum & al., 1992) directed by Prof. Burghes (University of Exeter, U.K.). The actual part of this project is involved in studying the development of students´ mathematical skills at lower secondary school level during two years in different countries. The first part of this project was carried out in Finland in the autumn 1994 by the authors. In this connection, the data of mathematical beliefs was collected, too.

Measures

The data measuring students´ mathematical beliefs constituted of students´ responses to six parts of a self-report questionnaire. These parts dealt with students´ beliefs about mathematics teaching (14 items), about solving mathematical problems (9 items), about mathematics learning (14 items), about their own activity in mathematics (8 items), and about their self-confidence in mathematics (12 items). The responses were given at a continuous scale ranging from -5 (fully disagree) to +5 (fully agree). The self-confidence measure was mainly adopted from Fennema & Sherman´s (1976) Confidence in Learning Mathematics Scale and modified to measure - as defined by Fennema & Sherman (1976) - students´ confidence in their ability to learn and perform well on mathematical tasks (e.g. "I think I could learn more difficult mathematics.", "I am not the type to do well in mathematics."). The rest of the items in the questionaire were adopted from earlier studies on students´ mathematical beliefs (e.g. Kloosterman, 1988; Pehkonen, 1992) or constructed for the study.

The part measuring students´ beliefs about mathematics teaching dealt with their views about a teacher´s and students´ actions during learning situations under the title of "Things Pertaining to Mathematics Teaching:" (e.g. "... teacher always carefully explains all things to pupils.", "... students solve mathematical problems together with other students."). Students' beliefs about solving mathematical problems were measured with items as "In solving mathematical problems, making mistakes is a sign of a low ability." or "In solving mathematical problems, there is usually only one correct solution.". Items on beliefs about learning concentrated in the part measuring students´ understanding of how you can learn mathematics (e.g. "You can learn mathematics only if mathematics teacher teaches well.", "You can learn mathematics by working hard for it by yourself."). The activity in mathematics problems dealt with students´ observations of their few self-regulative (metacognitive) actions or knowledge in the face of a problem, and was measured by items as "When I am about to solve a difficult mathematics problem I always carefully map out, how to do the problem." or "I know, what kind of mistakes I usually do in mahematics problems." The executed factor analysis mainly supported the partial scale construction.

Mathematical performances were measured by using the sum of the scores in four mathematical tests (Potential Test - 26 tasks, Number Test - 47 tasks, Algebra Test - 31 tasks, and Geometry Test - 20 tasks), which resulted in maximum 176 points. The problems of each of these tests of the Kassel project (Blum & al., 1992) were organized according to increasing difficulty, so that most of the problems especially in algebra and geometry tests remain unsolved by the students. The total sum of scores was then clearly below the maximum scores.

Results

Structure of beliefs

The first question of the study concentrated on the main factors describing students´ mathematical beliefs on the basis of the data, i.e. which were the principal themes in students´ mathematical belief structures included in the self-report questionaire. With all the items of the questionaire the factor analysis based on a scree-test resulted only three factors indicating students 1) active participation and effort in mathematics learning, 2) self-confidence in mathematics learning, and 3) external view of mathematics and mathematical tasks (in this order). This belief structure accounted only 24.8% of the total variance.

Based on the structures included in the questionnaire as well as on the eigenvalues of the factors, a more detailed description of the data was constructed. After excluding altogether 12 ill-behaving items from the scales, the factor analysis revealed 6 factors that were consistent with the views of earlier studies and that accounted 40.9% of the total variance. The factors were named 1) self-confidence in mathematics, 2) effort and regulative orientation, 3) external view of mathematics, 4) teacher dependence, 5) clear self-image in mathematics, and 6) co-operative learning (in the decreasing order of significance). The given factor solution reflected important features suggested to explain students mathematics learning actions and achievements through their motivational dynamics. The first three factors accounted together 70.2% of the explained variance (which was 40.9 % of the total variance) and the first factor - self-confidence - alone accounted 33% of the explained variance.

In accordance with the obtained six factors, six variables was constructed as sums of the item scores included in each factor, the Cronbach coefficient ranging from 0.736 (for self-confidence) to 0.399 (for co-operative learning). The correlations of these variables are given in the table below.

Variable	1	2	3	4	5	6
1. Self-Confidence	1					
2. Effort-Regul.	.299***	1				
3. External view	-.199***	-.132**	1			
4. Teacher depend.	.117**	.356***	-.098	1		
5. Clear self-image	.083	.222***	.161***	.123**	1	
6. Co-operative l.	.003	.17***	-.005	.179***	.127**	1

Table 1. Correlations for the given six variables of mathematical beliefs (N=453, and *** are for the error p<.001, ** for p<.01).

High self-confidence in mathematics correlated positively with high effort expense and regulative actions, with teacher dependence, and negatively with the tendency for external view of mathematics. High effort expense and regulative actions again correlated positively with teacher dependence, clear self-image and preference for co-operative learning, but negatively with the tendency for external view of mathematics. Moreover clear self-image correlated significantly with high tendency for external view of mathematics and teacher dependence, and preference for co-operative learning correlated positively with teacher dependence.

The correlations between variables were thus positive with the exception of tendency for external view of mathematics having negative correlations with self-confidence, and effort expense and regulative actions. One of the variables – effort expence and regulative actions – was significantly connected with all other considered beliefs, indicating that it has important mediating role among the variables.

Influence of beliefs

The second question of the study concerned the role of the given factors as predictors of students´ mathematics achievements. The correlational study indicated highly significant ($p<.001$) positive correlations between high score of mathematics achievements and self-confidence, and high effort expense and regulative actions. And tendency for external view of mathematics correlated negatively ($p<0.01$) with the scores. These results were confirmed through the regression analysis of mathematics achievement scores ($r^2 = 0.084$) with all the six variables as predictors of students´ mathematics achievements. The stepwise regression analysis resulted in an equation of only one predictor - self-confidence - with ß-coefficient 0.29. The bringing of effort expense and regulative actions together with external view of mathematics to the regression equation did not reduce this predictive value of self-confidence compared with other variables.

Instead a stepwise regression analysis of the same variables based on the factor scores ($r^2=0.099$) indicated also tendency for external view (ß= -.151) and effort expense and regulative actions (ß=.102) as predictors for students´ achievements after self-confidence (ß=.257). Thus as shown in the table of correlations, both of these two first features have slight predictive value for mathematics achievements. Yet students´ self-confidence in mathematics seems to be an efficient predictor of mathematics achievements compared with these two sets of mathematical beliefs, even if these two features correlate significantly with the achievements. This predictive value of students´ self-confidence in mathematics remains even in using the factor scores of the three factors produced by the scree-test in the stepwise regression analysis ($r^2=.094$). The factor of self-confidence is then clearly a better predictor (ß=.265) of mathematics scores than the chosen second predictor (ß=-.133) labelled as an external view of mathematics. Further, the first factor of the three given factors indicating students´ active participation and effort in mathematics learning, became rejected from the regression equation. Thus in addition to students´ self-confidence in mathematics as a clear predictor of high mathematics achievements, also students´ tendency for external view of mathematics and mathematical tasks can be seen to predict low mathematics achievements.

Conclusions

The study presented revealed important aspects of students´ mathematical beliefs systems in secondary school: students´ confidence in one´s ability to learn and do mathematics, an external orientation based on viewing mathematics as specific and fixed ability oriented school subject, and a uniform efficiency in mathematics and orientation to mathematics learning situations with effort expense and self-regulatory behaviours. From these belief structures students´ self-confidence in mathematics appeared as the most powerful predictor of mathematics school achievements. Since students´ confidence in mathematics has been noticed as one of the main factors affecting willingness to study mathematics and persist in mathematical learning performances as well as the arousal of affective responses during learning (Fennema, 1989; McLeod, 1989), there are significant consequences included in the levels of students´ self-confidence in mathematics. This influence also appear as an important discriminator between girls' and boys' mathematics achievements (Fennema, 1989) which also can be seen in the highly significant difference (p=.0001) between girls´ (-2.076) and boys´ (8.245) means of confidence and in the difference (p=.011) in their achievement scores of this study, both beneficial to boys.

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PRESERVICE SECONDARY MATHEMATICS TEACHERS' BELIEFS: TWO CASE STUDIES OF EMERGING AND EVOLVING PERCEPTIONS

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Abstract

Case studies were conducted to identify, describe and compare two preservice teachers' beliefs about teaching and learning mathematics during their year in a teacher education program. The research framework used Ernest's (1989) model to specify the beliefs under investigation. Repertory grid interviews were conducted to elicit emerging perceptions as the program began, and evolving perceptions eight months later. Other data included responses to a beliefs instrument and observation field notes. Subjects shared three emerging perceptions: teacher involvement/management, degree of student independence and communication. They shared three evolving perceptions: student centered learning, student active involvement in learning, and classroom management.

Introduction

Both the *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 1989) and *A National Statement on Mathematics for Australian Schools* (Australian Education Council [AEC], 1991) proposes that problem solving, communication, reasoning, and mathematical connections provide a foundation for the development of mathematical power in all students. The standards call for reform of the curriculum and for changes in its delivery. Classroom teachers have now been charged with choosing suitable mathematical tasks, with engaging students in active mathematical investigations and with the introduction of appropriate technology into the classroom.

This is a new area for today's serving and preservice mathematics teachers, the majority of whom have probably learned mathematics in situations where they were passive receivers of decontextualized facts and procedures passed on by their own teachers. Beliefs about teaching and learning, developed from successful learning experiences in such situations, present serious impediments to the changes called for in the reform of mathematics education (NCTM, 1989; AEC, 1991).

The research reported here examined the beliefs of two preservice teachers as they began a teacher education program. Investigation of teachers' beliefs has become a significant research endeavour in recent years (e.g. Clark, 1988; Clark & Peterson, 1986; Pajares, 1992). Studies of preservice teachers' beliefs conducted within and across disciplines suggest that as students enter teacher education programs, they do possess established although incomplete conceptions of teaching (Mertz & McNeely, 1991). Prospective teachers generally believe they will be successful, with judgements based on social and affective perceptions: Teacher education students express their enjoyment in working with school students and believe that they can relate to them (Weinstein, 1989). Research has shown that prospective mathematics teachers possess considerable knowledge of the general behaviours of mathematics teachers – such perceptions are often rooted in school experiences, including memories of past teachers, methods courses, and preservice teaching experiences (Mertz & McNeely, 1991).

Purpose of the Study

The study reported here utilised a naturalistic paradigm (Lincoln & Guba, 1985) in which the researcher was a participant observer. The aim of the study was to identify and describe the perceptions of teaching and learning mathematics expressed by two preservice teachers as they participated in the initial year of a mathematics teacher education program. The study compared the perceptions that emerged as the two teachers began program involvement with the perceptions that evolved after eight months in the program. The term "perception" is used here to describe characterisations of patterns of beliefs held by the two preservice teachers.

The study is primarily descriptive in nature and offers insights into possible transitions in preservice mathematics teachers' beliefs about what is central to the task of teaching. In addition, the methods employed hold promise as a way to document the evolution of teachers' beliefs in programs that explicitly attempt to challenge existing beliefs.

Theoretical Framework

While wide-ranging conceptions of teaches' beliefs are prevalent among educational researchers (Kagan, 1990; Pajares, 1992; Thompson, 1992), some characteristics of teachers' beliefs are said to be held within organised systems of the mind. Although beliefs may play an active role in the thought processes of teachers (Clark & Peterson, 1986), the beliefs themselves are considered through structures stored in the mind (Ernest, 1989). Within an organised system of beliefs, certain beliefs follow from others, beliefs vary in how strongly they are held, and beliefs become clustered in order to prevent ongoing confrontation among those which are in conflict (Green, 1971).

Another characteristic of teachers' beliefs is their disputability. There are no generally agreed-upon standards for evaluating beliefs. Knowledge may be justified as true through objective proof or by consensus of informed opinion, but such standards hardly apply to beliefs (Grossman, Wilson, & Shulman, 1989). One preservice teacher may believe that checking homework when class begins is the most effective way to assure that students have completed it; another preservice teacher may dispute that belief, claiming that it is more effective to collect the work at the end of the period. Both views may be regarded as equally valid.

As the example typifies, beliefs are highly personal in nature. Beliefs are often grounded in vivid singular experiences stored in episodic memory (Nespor, 1987). The highly personal nature of beliefs suggests an association with the affective domain, rich in feeling and subjectivity (Ernest, 1989). In that sense, one may conceive of beliefs as dynamic. Individuals continually restructure belief systems as they compare their present beliefs with their ongoing experiences. At the same time, beliefs can be very resistant to change due to the lack of standards for evaluating them. Logical argument may convince one of the truth or falsity of possible knowledge, but without direct experiences to the contrary, beliefs can be difficult to change.

These three characteristics of beliefs - their organisational structure, their disputability, and their highly personal nature grounded in experiences - helped to provide a framework for this research and to interpret its results. A model by Ernest (1989) was used to specify the beliefs under investigation.

His model includes beliefs as a component of mathematics teachers' thought structure. It stipulates four elements of teachers' beliefs: (a) a teacher's conception of the nature of mathematics, (b) a teacher's model for teaching mathematics, (c) a teacher's model for learning mathematics, and (d) a teacher's general principles of education. The researcher used these elements to conceptualise components of teachers' beliefs to be identified, described, and compared in the study. Particular attention was given to teachers' models for teaching mathematics, described by Ernest as: "their conception of the type and range of teaching actions and classroom activities contributing to their personal approaches to the teaching of mathematics. It includes mental imagery of prototypical classroom teaching and learning activities, as well as the principles underlying teaching orientations" (Ernest, 1989, p.22).

Methodology

The research setting was a secondary mathematics teacher education program at a university in Perth, Western Australia. As the 18 pre-service teachers began the program, the researcher collected and analysed data in order to select two subjects who were broadly representative of the group. Each of the 18 participants responded to a belief instrument (Van Zoest, Jones & Thornton, 1994) and completed a semi-structured interview to follow up on responses to the instrument. The Likert-type responses to the instrument were aggregated and the interviews were taped and transcribed. To select the research subjects, the researcher analysed the data to avoid selecting two subjects with very similar responses. The researcher used biographical information to further compare the potential subjects and finally selected a male (Ken) and female (Denise). The two participants entered the one-year program with no previous studies in education; each had already earned an undergraduate degree in mathematics. The program activities included a fixed sequence of courses as well as extensive field experience.

Data Collection and Analysis

Generating emerging perceptions. Upon selection, the researcher engaged each of the two subjects in a two-part repertory grid interview (Cronin-Jones & Shaw, 1992; Kelly, 1995). The researcher asked each subject to imagine teaching in an ideal mathematics classroom setting and to describe what the description, what were to become the *elements* of the repertory grid were recorded. The researcher then asked the subjects to describe *why* those events, behaviours and activities would be occurring. These descriptions generated the *constructs* of the repertory grid. Subjects then completed a two-dimensional grid on which they indicated their feeling on the relationship between an element-construct pair (1= no relationship; 2= neutral relationship and 3= definite relationship). Principal component analysis with varimax rotation (DeVellis, 1991) was applied to each subject's grid, identifying groups of construct statements for each subject. Each subject was asked to state a word or phrase (a perception) to characterise the relationship among the constructs in a group, and the researcher studied the data to capture the perceptions each subject identified.

Monitoring subject's perceptions. Data collection continued throughout the following eight months in order to monitor the development of each subject's perceptions.

Generating evolving perceptions. After eight months of program involvement, each subject again completed the two-part repertory grid interview with the researcher. This was conducted and analysed in the same way as the previous procedure, and the results used to identify evolving perceptions in the subjects' model for teaching and learning mathematics.

Comparing perceptions. Two levels of perception comparison were conducted by the researcher. First, Ken and Denise's emerging and evolving perceptions were compared separately. Similarities and differences were noted. Second, a cross-case comparison was made – Ken and Denise's emerging perceptions were compared with each other as were their evolving perceptions.

Results

Analysis revealed that the subjects shared three emerging perceptions: teacher involvement/management, degree of student independence, and teacher-student/ student-student communication. These perceptions, identified by the researcher as subjects began the mathematics teacher education program, tended to emphasise structural components of the classroom as well as affective factors. Subjects stated preferences for using formal classroom settings and for organising and managing a classroom. Teacher authority was settings and for organising and managing a classroom. Teacher authority was paramount with little emphasis on independence among students. Teacher-student communication was to be encouraged, but no mention was made of student-student communication. Differences in emerging perceptions included Ken's view of students' need for advance organisers in learning and Denise's recognition of being in transition from student to teacher.

The subjects shared three evolving perceptions: a focus on student learning, students' active involvement in learning and classroom management. These showed a shift in emphasis and perspective from the emerging perceptions: there was less commonality across perceptions, and the evolving perceptions that shared elements with the emerging perceptions- management and student independence - were identified from a different perspective. Ken and Denise's emphasis shifted to preferences, intentions, and concerns provoked by experiences with secondary school students. Differences in evolving perceptions included Ken's attention to student responsibility and attention to assessment, and Denise's regard for the classroom environment.

Discussion

Ken and Denise's initial program field experience and methods-course work represented their induction to mathematics teacher education. They had already completed a mathematics degree. Never before had their education explicitly focused on teaching and learning secondary school mathematics. It was a new perspective with which subjects entered secondary school classrooms and worked to fulfil methods-course requirements. The study sought to identify and describe subjects' beliefs about teaching and learning mathematics that emerged from this new perspective as well as to compare emergent beliefs with those that surfaced after eight months of program involvement.

The subjects' perspectives mirrored their program involvement with secondary school students. When the emerging perceptions were identified the subjects were, for the first time, entering school classrooms with the perspective of a teacher. Structural and affective stimuli were most apparent to the subjects. They attended to desk arrangements, grouping of students, student behaviour, and the ways classroom teachers related to their students. Eight months later, the subjects had long ago acquainted themselves with a mathematics lessons to secondary school students as required in their field experience. Other stimuli were more apparent at this time, such as a need to generate reasons for teaching and learning the topics they were responsible to help students learn. This suggests that the context and focus of the field experience had significant impact upon the subjects' beliefs and perspectives

The case studies show that during the preservice teachers' initial year of program involvement there was an evolution in their perspectives on teaching and learning mathematics. The study found that subjects' beliefs about what was central to the task of teaching were in transition.

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ON THE NOTION OF FUNCTION

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Abstract

This paper concerns certain topics about functions and potential problems students might have with them. The focus is on the more creative aspects, e.g. identifying, forming and using functions, rather than analysing given functions. The statement of the Fundamental Theorem of Calculus is used as a running illustration of many issues brought in.

1. Introduction

This essay deals with the concept of function. We shall be fairly broad in our discussion, and although much of it will concern functions in connection with real calculus we shall also consider the concept at its most basic and general level. To broaden our scope, in fact, we will also touch on the topic of functions in elementary abstract algebra.

Much literature on functions has been produced by the community of Mathematics educators. The majority of this research, however, concentrates on educational issues concerning properties of functions in the context of the Calculus. The result is not only a restriction of the types of functions looked at but also gives a stress on functions given in some explicit way (i.e. either algebraically or graphical). This tendency reflects the way that functions usually appear in secondary schools. This may be unfortunate, in that the real power behind the concept is in identifying functions and constructing functions for particular ends, and this source is largely untapped when students leave secondary school. This paper will examine some issues concerning functions in this particular light.

2. The Problem Illustrated

The applicability of functions is extraordinarily flexible. The definition of a function, when historically it was finally decided what it should be, was deliberately made such that this would be true. We shall discuss this further in the next section. The problem seems to be that most students never gain an understanding of this flexibility. We illustrate the above by considering the statement of the Fundamental Theorem of Calculus.

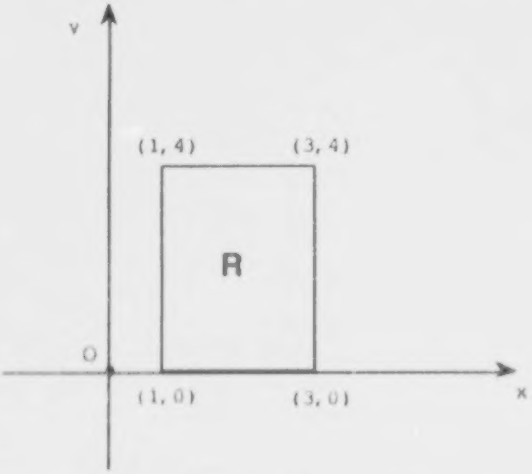
The Fundamental Theorem of Calculus, of course, involves the equality of two functions. Let f be a real, continuous function and let a be a fixed real number. Then the function given by the area under the graph representing f from a to x equals the antiderivative of f which has the value 0 at a. The theorem usually is expressed more symbolically and less descriptively than the above statement, which makes the functions involved seem less explicit. However they are no less present.

The theorem is of great interest as regards to examining how developed is the concept of function in students minds. Both functions need imagination to conceive of, and require a jump in abstraction to equate the functions (forgetting about "meanings"). This will be discussed more in section 5.

Research findings in [6] suggest that very few students have a conscious comprehension of the significance of the Fundamental Theorem after studying it. It was conjectured that a major cause was that the two functions which comprise the main proponents in the Theorem were somehow not appreciated. In the case of the area function, this was reinforced by asking a group of students, under examination conditions, the following question:

The question

For every $x \in R$ l_x represents the vertical line in the plane passing through the point (x,0). Let f(x) be the area of the part of the rectangle R which is to the left of l_x Write down a formula for f(x). What is its domain and image?



Only 22 out of 71 students attempted the question. Only 7 scripts obtained the key component of the algebraic expression, none gave a fully correct answer.

This research (and others, e.g. [7]) gives convincing evidence that most students are not able to conceive of such functions formed by areas under curves. We shall not attempt reasons here, but in later sections we shall hint some. For now we simply make the conjecture that a contributing factor is that students generally lack the background to feel comfortable and to work with functions which are given descriptively rather than algebraically. It is then reasonable to assume that the students are far from having facilities to identify and to use functions, facilities which are so basic and commonplace in high level abstract mathematics.

3. Formal Definition against Informal Descriptions

Definitions (of conceptual weight) always should inspire inquiry. Could I have expected this definition from my previous mathematical knowledge, or my intuition? Do I really understand why the definition has the exact form it does have? Students do not have the culture to analyse definitions in this way, and this contributes to the common sentiment that "in Mathematics you can prove anything". Students tend to categorise definitions into "understood" and " not-understood" (when often it is more fair to say that the true case, with some mental effort, is "partially understood").

The general formal definition of function, based completely in Set Theory, is certainly out of reach for even a reasonable understanding for school level students. A more informal description has to replace it. This is a trade off; our description will seem closer to the naive expectation that a function is a correspondence, but we lose sharpness into exactly what a function is (rather than what it involves) and we run the risk to make functions seem more "procedural" than desired. (The question of what a function is in informal terms causes even many teachers discomfort, see[8]). However with reservations, one might finally agree with [10] that the formal definition is of purely technical interest.

But it does not matter how we define or initially describe functions, the mathematical motivations to introduce such objects remain the same. We have to return to the questions with which we began the section. In [9], these motivations are studied and found to be vastly complicated. There is no reason, given the complication, that students immediately gain any insights into usage of functions. Students learn about functions through examples (even more crucially than in other topics). If the majority of examples of functions given in the class are formed from algebraic expressions, these then become the only bona-fide type of functions. The concept of function is very sensitive to personal experience, and it is probably only Mathematics graduates who ever sense its whole significance. If we want students to appreciate, say, the semantics of the statement of the Fundamental Theorem of Calculus, a stimulus must be introduced in earlier teaching to prematurely broaden the sense of function. The flexibility needed by the student to feel comfortable in identifying functions given descriptively might require a substantial teaching program.

4.Graphing: Pros + Cons

Historically speaking, the evolution of the concept of function was largely influenced by the evolution of real Calculus. The elementary concepts of real Calculus were motivated by considering curves in the plane; the rôle of functions was to find a more mathematically valid description of these curves. The task then was to decide on the most appropriate definition for this aim. (Which was tantamount to deciding which types of curves are best thought of as basic for consideration). Not surprisingly there were animated arguments as to what exactly the definition of function ought to be. It was only when it was realised that many theorems of real Analysis could be extended to other metric spaces that the more general notion of function became widely accepted. For more details of this history, see Boyer [2].

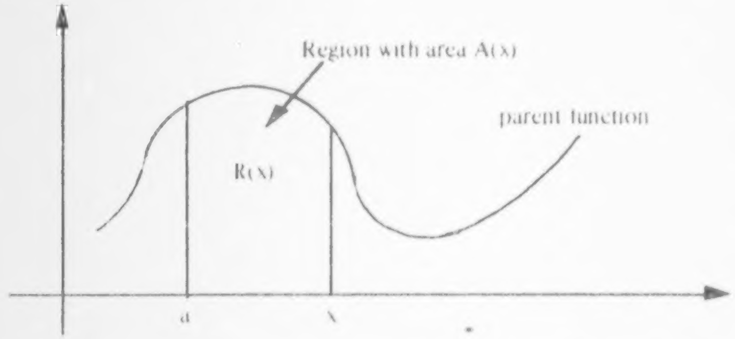
Within the context of real Calculus / Analysis, the emphasis on graphing functions is somewhat natural. However the practice does rather reverse the historical motivation, in that the graph is drawn in response to a usually algebraically expressed function. At school, graphing may seem rather academic, as you rarely use the graphs; they are just an end to themselves. They constitute a convenient mode to summarise some properties of the given function obtained by algebraic analysis. It is uncommon that, in the other direction, students are encouraged to use common sense or informal reasoning to deduce properties of functions from their graphs (for a simple example to deduce that between two local maxima of a differentiable function with domain **R** there must be a local minimum). This is a pity , as this is much more fruitful course to broaden experience in using functions (though, still, in a rather limited context). Eisenberg in [3] identifies and bemoans the apparent reluctance of students to visualise when thinking about functions; this is partly due to the above mentioned trend in the school syllabus.

In a more general context, it is more questionable whether the graph should be preferred to any other representation of a real function. However the legacy left from the Calculus tack obviously places the graph in a privileged position. In fact students, who may lack the sophistication involved in distinguishing a "representation" from the original object, may confuse the graph with the function itself (this confusion would be compounded if the students are exposed to a definition of function involving ordered pairs). The fixation on the idea of the graph may well impede the facility to recognise and accept functions which can be formed naturally from other geometric parameters or descriptions. (An obvious example would be from polar co-ordinates instead of Cartesian co-ordinates). It is interesting in this light that in 7 out of the 22 scripts who attempted the question described in section 2 there was evidence of some confusion between f (function of area) and the constant function 4 over [1,3] whose graph is suggested by the given diagram.

As an aside, graphing also has more practical problems. Not only is it difficult to "do" even quite simple "operations" of functions from their graphs, it is confusing to understand exactly what you are doing; are you operating with functions or are you operating with transformations of the plane? In this way, choosing your standard representation of a real function as an imbedding in the plane may be unsatisfactory. Inspired by this and other considerations, another more "dynamic" (computer aided) representation of real functions was suggested by [4] for exposure to school students. In short it involves two separate "real lines", one representing the domain, the other representing the codomain. A cursor on the domain line is moved which controls another cursor on the codomain line according to the given function. This certainly seems to give a more honest description of what a function is on the most basic level, and the authors are optimistic of the potential of this teaching tool: "...we came to believe that this and other dynamic representations might have potential to help the student to objectify the function concept, provide a much-needed foundation for the techniques of constructing and interpreting a variety of graphs, and even develop in students a broad and flexible concept of function, including functions whose domain and range are not sets of numbers". In short, answering a lot of the concerns of this essay!

5. Questions of Meaning and the Rôle of Variables

Functions extracted in a descriptive way from, let us say, a geometric configuration, cannot preserve any meaning from the context from which it was conceived. Functions essentially are abstract objects. This may be difficult for students to accept. Let us consider the two functions (the "area function" and the antiderivative) which are the main "players" in the statement of the Fundamental Theorem of Calculus, (see section 2). These two functions are formed from an original, unspecified function which we term the "parent function": we assume that the parent function is continuous on **R**. Of the two functions, the area function is more directly formed from the parent function but retains a strong linkage with regions in the plane.



The antiderivative is less directly formed from the parent function (in fact to the extent that it is far from obvious that it exists). It is an hypothetical function which is related to the parent function through a property (differentiability) of the hypothetical function. This may seem perverse in that we seem to be constructing the "known" function from the "unknown" function. To understand how we can accept that the antiderivative is in fact formed from the parent function (rather than vice versa) we have to move on the conceptual level from the "constructive mode" to the "existential mode". (This, we maintain, illustrates a general shift of vast importance for comprehending the potentiality of using functions, but the same shift perhaps is a fair candidate for the best description in a few words of the distinction between naïve and abstract mathematics. As such, this shift is gradual and difficult).

Because of the indirect way the antiderivative is obtained, any meaning attributed to it is not going to be sharp. However, tentatively, tangents of the antiderivative can form some reference (as for every x the slope of the tangent of the antiderivative at x equals the value of the given parent function at x). Hence both the real functions (the area function and the antiderivative) do have references to geometric objects which can contribute to the feeling that they have "meanings". This brings to the fore the definition of equality of functions. Were these two functions considered intrinsically connected with their geometric references and that the definition of equality of functions was not made explicitly, there might be a tendency not to acknowledge the equality proposed in the Fundamental Theorem on the grounds that we are trying to compare incompatible geometric objects (i.e. regions and tangents). This illustrates a crucial motivation for the development of the notion of function; its abstract nature (with what seems at first a rather empty definition of equality) allows working in the same framework constructs from completely different contexts, at the same time enabling associations to be retained if desired.

The Fundamental Theorem, being a general theorem, does not allow the reducing of the two functions into specific algebraic expressions which might be more acceptable psychologically to equate. However even if we take a specific case the reduction may involve some epistemological problems. Let us illustrate with the area function f in the question in section 2. For any value of $x \in [1,3]$ we may show that $f(x)= 4(x-1)$. Hence $f(x)= 4(x-1)$ for all x in the interval. The use of the variable x may seem mundane here, but the way it enables the consideration of the particular to become simultaneously the consideration of the general is in fact quite subtle. (It bears analogy to the switching between thinking a function as a process and as an object). If students do not appreciate the rôle of variable they are seriously handicapped in working with functions.

Variables clearly are very important in influencing the notion of function. Their "neutral" character is difficult to accept, and the natural feeling that variables are an integral part of a function may bring various problems. Some research has been conducted on variables (e.g. [1], [11]), but the author feels there is still a lot to examine.

6. Functions on Finite Sets

A function comprises not only a correspondence but also the two sets forming the domain and the codomain. It is the structure of the domain and the codomain which largely determines the ways that the function might be used. The structure of the real numbers is very complicated and so the issue of how real functions can be used would naturally be also complicated and elaborate. To obtain breadth in the notion of function surely functions with domains of other types (i.e. domains not containing an interval of the reals) should be considered. For simplicity finite sets would seem most profitable, as we might hope the simplest domains / codomains would give the most fundamental insights into the concept of function.

If we were given two finite sets (neither with internal structure, e.g. operations) the characterisation of all functions having the one set as its domain and the other as its codomain is fairly straightforward. The task of enumerating these functions is a reasonable test to investigate whether a student understands the situation. Similarly if the same finite set (with *n* elements) was both the domain and the codomain, the bijections are easily identified and enumerated. Up to now, we seem to have little of real depth. However if we make a slight abstraction by thinking of the bijections as objects, composition as a binary operation, the set of bijections S_n is one very much with structure, and invites many questions. For instance, one might attempt to identify all subsets of S_n that are closed with respect to composition and inverses. This problem in general is very difficult (if we were to vary *n* over the positive integers, the problem would include identifying every finite group up to isomorphism). We see that our apparently limited situation of considering bijections on finite sets leads fairly naturally (though not perhaps intuitively) into one of the most rich branches of abstract algebra.

Isomorphism of groups (and other classes of mathematical objects) are very interesting vis- a - vis in understanding what functions can do. Isomorphisms form mediums to allow different objects in the same class to be considered the same if they have consonant structure. The definition of isomorphism (which differs for each class of mathematical objects) specifies exactly what of the structure is relevant; the rest of the structure may be forgotten. This provides a basis of working on all objects which are isomorphic as one. Isomorphisms, and other types of functions in other guises, can be seen as devices allowing the mechanisms of Mathematical reasoning to run smoothly.

What is noticeable is that functions on finite sets seem to have a different feel than real functions, but, against expectations, the issues are probably as deep. Despite of this, little research in educational circles seems to have been done in this direction. However see in [5] some fieldwork on isomorphisms of finite groups.

7. Final Remarks

As a student progresses from higher secondary level to tertiary level mathematics, he / she will become more and more dependent on a good "sense for functions" to understand the statements of theorems (and even more their proofs). In this way, the statement of the Fundamental Theorem of Calculus may be regarded as a landmark in that many subtle considerations come in as this paper has shown. As students often fail to comprehend this theorem, it seems their notion of function is lagging behind their needs at the time.

Corrective teaching action seems in order, but this aim is complicated by the fact that the notion of function is very dependent on context, so it is inevitable that much of the enrichment of the notion has to eventually rest on the shoulders of the student himself / herself. However attempts could be made to illustrate the main issues by choosing suitable contexts to boost the process; this might be served simply by stressing and discussing explicitly the rôles of functions in topics that are already taught. The main issues would include many concerns brought up in the paper: not to be too influenced by graphs, to be flexible in identifying functions in different contexts and in forming functions from others, how to treat meanings and how to use variables, and to understand how functions are often specially designed as tools which allow Mathematical discourse and theory building.

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Reasoning geometrically through the drawing activity

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Abstract. The *role of drawings, and in particular the activity of drawing , is inspected in order to focus the main functioning of drawing in the construction and evolution of geometrical reasoning on the sun shadows phenomenon. The discussion is based on the analysis of some protocols taken from an experimental activity at the 5th grade level and is inspired by the theory of figural concepts (Fischbein, 1993).*

Introduction

This paper discusses the role of drawing in the geometrical approach to the sun shadows phenomenon. The experience we refer to is part of a long standing project, in which "An everyday life field of experience is worked away for a long time and the work is basically directed by the requirement of the development of the knowledge concerning the field of experience itself...." (Boero et al. 1995, p. 156)

In the reference frame of this project, previous investigations highlighted some aspects related to drawing. For instance, "the importance of the sign systems proposed by the teacher in order to stimulate the transition to a scientific conception of the phenomena" (ibid., p. 158) and more generally, that "the "shadow diagram" seems to modify even the way of thinking the relationship between the height of the sun and the length of the sun shadows"(Boero et al. 1995, p; 159).

The following analysis aims to go further and investigates the role of drawing in the evolution of meaning of geometrical modeling.

Physical space, drawing and geometry

Geometry is deeply rooted in everyday life experience of physical space with which maintains a complex link. The complexity of these connections corresponds to the complexity of teaching and learning; unfortunately, too often the relationship between physical space (generally referred to as space) and the theoretical domain is not questioned and the move from observation to theory is considered as being natural. Despite the undeniable link between geometry and experience, a sharp distinction must be made between the ideal, abstract space of geometry and the space where all experiences are accomplished. In this complex frame of reference what is the particular role played by drawing?

From a general point of view, a drawing has an intermediate position between concrete reality, of which it participates, and the abstractness of geometry, of which it can be a representation(Gonseth, 1936).

In order to clarify this point, it is useful to refer to the theory of figural concept, as it was introduced by Fischbein (1993).

Although geometry is a theoretical construction, from the psychological point of view geometrical concepts conserve *in the reasoning process an objective, pictorially representable property of reality which is* ***space*** .

In this sense and for this reason, geometrical figures are mental entities which possess, simultaneously, both conceptual and figural properties. *A* **figural concept** *is then a mental entity which is controlled by a concept, but which preserves its spatiality* (**Fischbein**, 1993). As a consequence, geometrical reasoning is characterised by a dialectic tension between the two components of geometrical concepts (Mariotti, 1991, 1993, 1994).

Let us consider a graphic representation, a *drawing*, of a geometrical concept; for instance, let us consider the drawing of a square. As soon as a square is drawn, it becomes a particular instance of the concept of square, sharing with it the figural component, but missing some basic characteristics of the conceptual component, mainly its generality. For that reason a drawing can be used productively in geometrical reasoning, only if it can be conceptually controlled. Drawings are particular and this fact may be an obstacle to a correct conceptualization, if generality is not completely attained. On the other hand, conceptualization is possible only through keeping the mental control of the drawing. Passing from 2-D to 3-D geometry, the mental process becomes more complicated, in particular when the relationship between the geometrical concept and its representation is ruled by geometry itself, as in the case of perspective (Parzysz, 1991), but still of the same nature.

A completely different problem arises when drawing is involved in a modeling process. In this case, a drawing has an intermediate position between the real world of which it represents some aspects and geometry of which it represents some concepts and relationships; the representing process must be articulated anew, relating the graphical representation at the same time to the physical and to the geometrical referent.

The aim of the following analysis will be to consider the status of drawing in the very specific situation of a geometrical model of a physical phenomenon. The ambiguous role of drawing may be determinant in triggering a dialectic interaction between the descriptive function of graphic representation and the modeling process, which leads to a geometrical interpretation of the physical phenomenon.

The Shadow Diagram in the teaching experiment

Let us consider the sun shadows phenomenon and its geometrical model. In the experimental framework of the Genoa Group project (see Boero et al., 1989) the sun shadows phenomenon constitutes a basic context, a *field of experience*, in which the mathematical modeling activity is developed. One of the crucial points is the evolution from the common sense conceptions of the pupils to scientific (in particular, mathematical) models of reality provided by the school culture. The "shadow diagram" represents a key element in the geometric model of the phenomenon, but previous investigations show how difficult is the mastery of this element and the weakness of its use in problem solving situations (Boero et al., 1995a).

Producing and illustrating a hypothesis

At the fifth grade level, a specific situation was set up in order to better analyse the functioning of the shadow diagram. After some activities on shadows and the introduction of the "shadow diagram", the following problem situation is presented to the pupils. At this point the "shadow diagram" and its characteristic elements are well integrated in the conceptualization of the phenomenon

> *"Two boards with two nails of identical length are horizontally placed one in the yard and the other on the terraced roof of the school. How are the "fans" of the shadows recorded at the same hours of the same day?"*

Pupils are asked to formulate a hypothesis about the length of the shadows in the two cases and to express their hypothesis through a drawing . Different hypothesis are given and according to these hypotheses different drawings are drawn. The "shadow diagram" is widely used; it appears a partial model of the phenomenon, with a descriptive function, both in the case of the correct and the incorrect hypothesis.

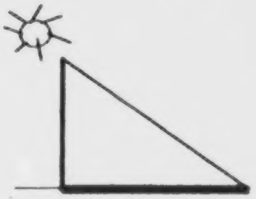


fig. 1 The "shadow diagram"

The protocol of Noemi is a good example in the case of the hypothesis about the difference of the shadows' lengths (see the annexe). On the other hand, in the case of hypotheses about the equality of the shadows' lengths , most of them seem to refer to particular situations, characterized by the presence of certain regularities; for instance, a symmetry in respect to the sun, as in the case of Sara's drawing (see the annexe); in this case, the global figure is well organized in an isosceles triangle. Generally, in this first approach to the task, the drawings are consistent with the corresponding hypothesis about the phenomenon and no conflict appears. The drawings simply accompany the verbal expression and aim to illustrate the hypotheses. The problem of representing the situation is solved by drawing the main elements, the sun, the nail and the shadow; but, instead of the actual spatial relationships among them, the drawing reproduces a contract, quasi symbolic, version of them. The representation is organized by a triangle but, according to the standard "shadow diagram", and in contrast with any geometrical consistency, it contains the sun too (see fig. 1). This is possible only through a symbolic code of representation, which overcomes the actual spatial relationships. The drawing seems to result from a dialectics between "knowing" and "seeing", which accomplishes (fits) the descriptive function but is not geometrically consistent, i.e. it is not yet a geometrical model. These kinds of representations, based on the shadow diagram, express the hypotheses, but they

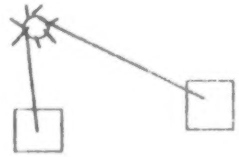
contain a potentiality for conflict, because of the intrinsic inconsistency of the spatial relationships among the elements that they represent: because of the presence of the sun the parallelism of the rays conflicts with the possibility of representing their origin in the sun .
As soon as a reasoning about the physical phenomenon will be transferred on the drawing, in other words a geometrical reasoning is attempted on the diagram, the required harmony between the figural and the conceptual aspect of the figure may reveal the inconsistency and conflicts can appear.

Drawing to explain

Soon after this activity, pupils can directly observe the phenomenon and realize that in both cases the shadows have the same length. The teacher presents the second task to the pupils

> "*Explain what you observed, use a drawing for your explanation, too.*"

As soon as the drawing must be used with the specific aim of explaining the phenomenon, conflicts may appear. Let us consider the following protocol.

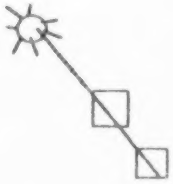


1° drawing

Stefano: "In my opinion, the fan would change, because if on the morning we move it to Pegli (a village to the east) the ray would reach the board earlier and the shadow would be shorter. On the contrary, if we move it to Voltri (a village to the west) the shadow would be longer because the ray would be more inclined and would go further."

After the observation of the phenomenon.

"Maybe, the shadows are equal, maybe, because in spite of the fact that the sun ray is in different points, it could reach both boards and the slope of the ray is equal and thus the shadows become equal".



2° Drawing

Teacher: "Can you explain to me why only one ray?

Stefano: "Yes, because, maybe, the boards were [turned] to the same direction and so were the two nails and thus the same ray reached them both. It is possible to verify it by drawing the two fans one over the other and see if all the shadows were one over
the other and see also if there were equal distances

Teacher: "Has been the sun always there?"

Stefano: "No ... now I understand that there are two different rays."

Stefano starts to draw. (Fig. 2)

After many attempts he says:

"I prefer to explain it by words, I cannot draw it" ... at the end he writes "In my opinion, I could draw the parallel rays and thus maybe, the shadows are equal ... "

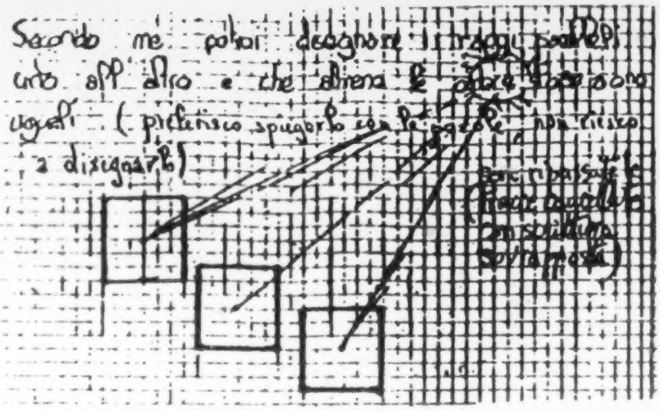

Secondo me potrei disegnare i raggi paralleli
uno all'altro e che forse le ombre sono
uguali (preferisco spiegarlo con le parole non riesco
a disegnarlo)

fig. 2

The verbal expressions Stefano uses, in particular the word "maybe" ("forse", in italian) suggest that the pupil considers the situation that he observed (shadows of equal lengths) a particular case in the general framework of the sun shadows phenomenon; thus, when he wants to give an explanation through a drawing, the particular case must fit into the standard representation, i.e. the "shadow diagram". At this point, the potential conflicts arise and Stefano does not succeed in his attempts. Stefano wants to draw parallel rays in the standard frame of the shadow diagram, where the rays converge into the sun; the inconsistency of the 'model' available is revealed.

Mentally, Stefano is able to conceive and perhaps he can even imagine the possibility of this specific situation; but, when he wants to transfer this idea into the drawing he cannot find the same coherence. In the drawing, as soon as the three elements (sun, nail and shadow) are represented so are their spatial relationships: the graphic representation and the real situation share some spatial properties. Thus, the drawing of the two "shadow diagrams" must determine the spatial relationship between the two shadows as a consequence. The request of producing an explanation inside the graphic context, forces to look for consistency inside the drawing, i.e. the drawing must become a geometrical model of the phenomenon.

Comparing the graphic and the verbal representation of the same situation, one realizes that the two modalities differ essentially. In the graphic mode, *the claim for consistency between the spatial properties is intrinsic of the nature of the drawing, which represents spatial properties through spatial properties.* In the verbal mode, the spatial properties of the representation (spatiality of the written words) are not expected to fit the spatial properties of the situation that they represent. As Stefano says:

"I prefer to explain it by words, I cannot draw it" ...

Verbally he can express the justification, but it is not possible to draw it.

It is interesting to remark that not everybody experiences the possible conflict; different levels of geometrical consistency can be required by pupils to their

drawing, so the drawings provided can represent different conceptualizations of the situation. For instance, the first drawing provided by Noemi (see the annex) presents perspective elements, which witness of a descriptive intention in this representation; the "shadow diagram" is inserted into the picture.
In the first picture, Noemi maintains the correspondence with the real situation and draws only one sun for both the triangles, in the second picture, the two triangles are separated and the conflict between the model of the shadow diagram and the situation observed is overcome moving to a symbolic level. In fact, the sun is drawn as part of the "shadow diagram", as a symbol to indicate that the diagram refers to the shadow phenomenon and certain segments are to be considered sun rays. Noemi does not refer to the sun, rather to the rays and says: " I drew the rays with the same inclination ..."

Discussion

The previous analysis shows the complex relationship among a drawing, a real situation and a geometrical model; it highlights the crucial role played by the graphic representation, both as a process and a product, in developing the meaning of geometrical modeling in relation to a physical phenomenon.
The theoretical frame of figural concepts suggests an interpretation of the solution process; in particular, the notion of figural concept focuses on the distinction between spatial description and geometrical reasoning on the model. The crucial passage is from the descriptive function to the explaining function of the drawing; in this passage the conceptual aspect concerning the geometry of the diagram must be logically co-ordinated to the interpretation of the phenomenon, so that the geometrical logic of the diagram must fit the interpretation of the diagram according to the phenomenon.
The need of consistency between the conceptual component and the figural component prompts a dialectic process which makes conflicts to appear and opens the way to a new relationship among the drawing, the phenomenon and the model.
The meaning for "geometrical modeling" may emerge through the interplay between interpreting and reasoning geometrically .
The spatial nature of the representation directs pupils' reasoning towards spatial considerations and helps to overcome conceptualizations of different nature like the following (see also Boero et al., 1995a).

Daniele (11 year old) : "... In my opinion, the shadow is the result of the fight between the two extreme powers, good and evil [...]. At a certain moment, the shadow is longer than us because the power of the devil is stronger and it is shorter when God, his power, is stronger than that of the devil." "

Generally speaking, drawing represents a mediation between reality and geometry, the role of mediation is accomplished in the passage from the descriptive function to the interpreting function, and corresponds to the change of status of the drawing from a *picture* of the real situation, resulting from a well adapted compromise between observation and knowledge ("the shadow diagram"), to a *model* (geometrical model) of the phenomenon.

The change of status is helped by the particular nature of the graphic representation, which in the case of spatial properties has an isomorphic character; that is, spatial relationships are represented by the *same* spatial relationships. For instance, the inclination between the sun rays and the nail are represented by the inclination between the lines representing the sun rays and the segment representing the nail.

From the didactic point of view, it is interesting to remark the role played by the teacher. The direct intervention of the teacher breaks the coherence reached by the pupil in his mind; pressing him to explain his thoughts by a drawing, the teacher makes the conflict appear. It is important to remark that what determines the change of status, from a picture embodying a knowledge to a model interpreting a phenomenon, is not only the drawing in itself, but also the fact that the meaning of the task suggests that the drawing becomes the working environment for elaborating an interpretation. Actually, the function of the drawing is derived from the specific task: using a drawing, first express a hypothesis and then, give an interpretation of the phenomenon.

The specific task introduces the idea of a **consistent** graphic description of the phenomenon, through which an explanation can be found. Reasoning on the drawing and looking for a logical explanation introduce the pupils to the modeling process; in particular, because of the spatial nature of the properties considered, to **geometrical modeling**.

While looking at the drawing (picture) and reasoning on the phenomenon, there is a shift to reasoning on the geometrical figure without forgetting its interpretation (geometrical model); looking for consistency inside the drawing (for instance, the presence of the sun becomes a problem as well the presence of two suns!) is at the same time looking for an explanation of the phenomenon and looking for a reasonable geometrical relationship among the elements of the figure, i. e. reasoning geometrically on a model.

Acknowledgements. *I thank Paolo Boero and Rossella Garuti for the passionate discussions from which this ideas arose.*

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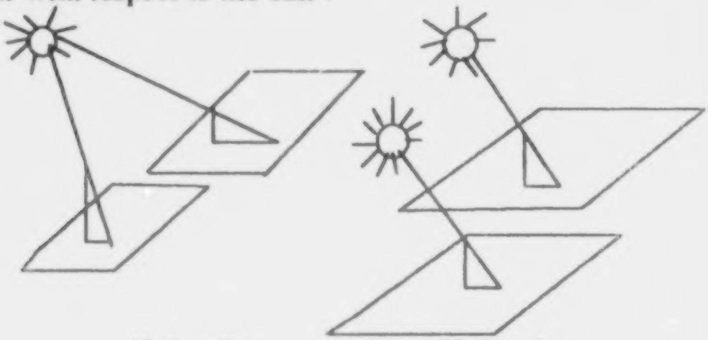
Annexe

Noemi: "In my opinion, the shadow changes because the inclination of the sun changes. The inclination of the sun changes because I can stay nearer of further with respect to the sun .



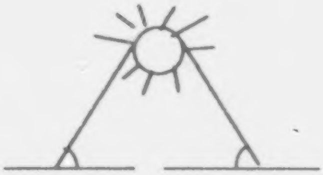
1° drawing 2° drawing

"I drew the two tables equal and both in the same place. In both cases, I tried to draw the ray with the same inclination and I saw that the shadow was equal. Actually, I wrote that it was different, because I thought that the table had been moved form right to left, actually one of the table was in the yard and the other on the terrace ..."

Teacher: "How did you draw the rays?"

Noemi: "I drew the rays with the same inclination , because we saw that, if the ray is more inclined it produces a longer shadow and when the ray is less inclined it produces a shorter shadow; thus in order not to have different shadows I drew the rays with the same inclination."

Sara: "In my opinion, if the table is moved the fan of the shadows does not change because when we were younger we observed the shadows, we spread in the yard and the shadows were all in the same direction. Thus the fan is the same."



Sara's drawing

THINKING ABOUT GEOMETRICAL SHAPES IN A COMPUTER-BASED ENVIRONMENT

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Despina Potari — **University of Patras**

This study investigates children's constructions of the concept of geometrical shape in a specifically designed computer-based environment. The focus is on the ways children relate figures with their properties and on the possible hierarchical relationships that they form. A constructivist teaching experiment is used to explore children's thinking. Four pairs of 11 years old children participated in this study and the results from one of the pairs are presented in this paper. In this specific context, children seem to use intuitive and dynamical models to built relationships between the shapes and their properties, and between different classes of shapes.

Introduction

Children's thinking about geometrical shapes has been studied by different researchers. These studies have mainly focused on children's ability to identify geometrical figures by relating concept's definitions and figures, and on children's conceptions (Wilson, 1986; Hershkowitz, 1989; Patronis & Spanos, 1991; Warren & English, 1995). The research in this area has also been greatly influenced by the work of Van Hieles as it appears in the review paper of Clements and Battista (1992).

A number of studies have emphasized the role of computers in teaching and learning geometrical shapes. In particular, LOGO environments have influenced the way children conceive geometrical figures (Kieran & Hillel, 1990; Clements & Battista, 1990; Battista & Clements, 1992; Kynigos, 1993). Laborde (1993) analyses the contribution of geometry software like Cabri and Geometric Supposer to considering functional and analytical aspects of geometrical objects. Dorfler (1993) supports the view that with computer tools, geometric figures, constructions, and systems of relationships themselves can become the objects of activity and are no longer just the products of the drawing activity.

In this study, a computer environment has been developed to help children link the properties of geometrical shapes with their visual images and form "figural concepts" (Fischbein, 1993). In particular, the program aimed to help children to construct hierarchical relationships of polygons within the constraints imposed by the use of software . The screen shows a straight line segment that the pupils can break to get a number of different pieces. They can rotate these pieces and form different polygons, all of which would have the same perimeter. Children can save their constructed figures and study their similarities and differences. A ruler for breaking the segment, the lengths of the pieces and the angles formed in each case are given if required. The idea of transforming a geometrical figure whose perimeter remains constant has mainly been used for the study of the relationship between area and perimeter (Montangero & Smock, 1976). Details about the development of software and its philosophy are given by Markopoulos & Potari (1995).

Briefly, we consider this computer program as a tool for organising a possible learning environment for the children by developing tasks relevant to the goals of the participants of the learning process. In particular, this study aimed to investigate, under the posibilities and constraints of this specific context:

- children's constructions of the concept of geometrical shapes, and how these develop
- whether and how children can relate figures with their properties and what kind of hierarchical relationships they build.

Methodology

To explore children's constructions we have used a "constructivist teaching experiment" (Cobb & Steffe, 1983; Steffe, 1991) where the researcher acts as a teacher who models children's constructions. The researcher interacts with the children by selecting tasks according to his/her interpretations of children's actions in each teaching episode. The researcher also studies the development of children's constructions over extended periods of time.

In this experiment we worked with four pairs of 11 years old children . Each pair worked on the developed software in a 45 -minute session per week for 10 weeks. The sessions were videotaped and transcribed. The pairs were selected from two fifth-grade classrooms of the same school according to their responses to a given task. Children were asked to draw three different quadrilaterals and give a written description to one of their friends about their drawings. Then they exchanged their descriptions and tried to draw the figures according to the descriptions they received. At the end, they compared their initial drawings with those produced from their written descriptions and gave a new description if they wanted. This way, we identified two groups in each classroom according to the way they could relate the figure with its properties. These groups seemed to belong respectively to the first two Van Hiele levels of thinking (Crowley, 1987). Two pairs were selected from the first group and two from the second. We also took into account the gender (5 girls and 3 boys) as well as the teachers' opinion about the personal relationship of the children.

The tasks used were based on our interpretations of children's actions. They varied from a simple construction of figures that children wanted to make to more goal directed tasks, where children were asked to make shapes with a common property; for example, to make different quadrilaterals with equal sides. In most episodes, we encouraged children to face extreme cases, e.g to make a triangle which can just be formed, and generalisations about the hierarchical relations of figures based on their differences and similarities.

Results

We present below our preliminaly results coming from our analysis of the teaching episodes of one group of children: Artemis and Lambrini. These two girls had been initially characterised as conceiving the geometrical figures visually. The tasks that these children faced concerned mainly quadrilaterals with the exception of two episodes which referred to triangles. The transcribed episodes were analysed to identify children's actions and to interprete them. The possible change that occured on children's actions, conceptions, justifications throughout all the episodes was also examined. We summarise below some points that arised from our analysis:

The appearance of the prototype phenomenon

The phenomenon of looking for "super examples" for a concept described by Hershkowitz (1989) as "prototype" was apparent particularly in the first episodes and especially for Lambrini. Children tended to draw regular forms in the upright position. In the case of triangles their first choice was an equilateral triangle but because of their difficulty to cut the segment in three equal lengths, the closest regular form that they chose was an iscoceles triangle with one of its equal sides horizontal. They characterised this triangle as " almost a triangle" . As early as the first episode, Artemis was aware that the different orientation does not change the figure but for her the location is a factor that makes things look different. She supported her opinion by imagining a transformation which was keeping the shape and its size unchangable "they look different but they will be the same if we rotate them". This conception is not generalisable in non-familiar shapes. This is evident in the subsequent episode where she doubts about the rigidity of an obtuse-angled triangle which she had constructed. Throughout the experiment, Artemis developed a different line of argument for supporting the rigidity of an acute- angled triangle. Now, she considers the angles of the triangle as a way to determine its rigidity: "these cannot come closer" (while showing the sides of an angle of the triangle). The developing of this awareness has been probably encouraged by the work done on quadrilaterals with equal sides where the different shapes were the result of the change of the angles. Moreover, Artemis can anticipate now the rigidity of the triangle without the need to experiment. She is certain for the result of her actions before acting: "No matter how we turn it, it is always the same".

On the other hand, Lambrini was not always certain about the rigidity of the triangle. When the change of the position was apparent, she considered the figures as different. Her image of the triangle depended on its orientation. It was a fixed image which could not rotate. Although Lambrini constructs the same shape in different locations on the screen, their static apperance dominates her conceptions. In subsequent episodes, Lambrini recognises that the triangle is the same when viewing it from different points but she tends to come back to her initial beliefs. The change of the context by the teacher: "imagine that you cut these two triangles and you place one over the other", or: " you stand up and then you lie down. Do you change?" seems to have helped her to consider that a different orientation does not necessarily imply a change of the properties of the figure.

Although we identified the existence of the visual - perceptual limitations described by Hershkowitz (1989), especially in the first teaching sessions, we had indications that the overall interaction between the children, the teacher and the tasks assigned to them based on the computer environment, helped children to overcome to a certain extent these limitations. The fact that, in the whole process, children were not given any verbal definitions for the particular shapes but they were encouraged to experiment, reflect and discuss the results of their actions, seems to have an effect on their initial conceptions and beliefs.

Relating figures with their properties

The program itself led children to consider the relationship of the sides in order to draw their chosen shape. For example, to make a square they had to cut the initial segment into four equal pieces. Although children had faced simple geometrical shapes and their properties in school, they used this knowledge mainly to recognise and name figures that they had constructed. The following example (7th episode) gives some indication of how children come to identify a parallelogram and how they relate it with other familiar shapes. In this episode children had already constructed different shapes with two pairs of equal sides (non- adjacent) and now they study an oblique parallelogram having the conventional orientation of a rhombus (figure 1).

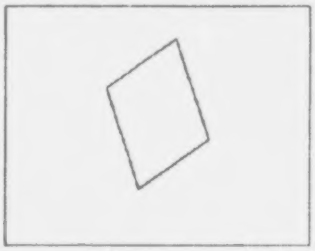

Figure 1

T: What is this figure?
L: Rhombus.
A: Yes, almost.
L: It does not have all its sides equal.
A: Almost, almost.
L: Almost. If we see it from this point, it is a rectangle.
Now if you look at it from the front, it is like a rhombus (1)
A: Yes, like a rhombus, something like that.
L: Yes, but it does not have all its sides equal.
A: Well, this is a rhombus but it does not have all the sides equal.
L: Yes, but then it is not a rhombus.
A: Right.

All of a sudden, Lambrini relates her school knowledge with the present situation.

L: It can be done. It is an oblique parallelogram.
A: Yes! Yes!
T: How do you know it?
A - L: We have done it with our teacher.
T: Why is it an oblique parallelogram?

A: Because the sides are equal... the two angles are equal and the other two are equal again.
L: Yes, and it slants like the oblique parallelogram.

The above extract shows that now even Lambrini considers the properties of a rhombus very crucial for deciding the type of the figure while in the first episodes her decisions were mainly perceptual. The last part of this extract shows how children can give meaning to their school knowlege, something that was not very obvious in the earlier episodes. Lambrini's comments in dialogue (1) above, could be interpreted as showing children's tendency to view figures in perspective but also an ability developed through involvement to consider things from different points of reference: "if you see it from this point... if you look at it from the front". Throughout the preceding episodes , this tendency was very strong for both children and it first appeared when children had constructed different quadrilaterals with four equal sides and they were asked to find out their differences. Both children were viewing the figures as objects in space, a conception that although it seemed to have changed in some cases, it was coming back again quite often (a similar case was reported by Clements and Battista, 1990).

Building hierarchical relations

The emphasis of the tasks on identifying similarities and differences of figures and the focusing on extreme cases, seem to have helped children to exhibit dynamical models similar to those found in young children by Gagatsis & Patronis (1990). These models helped them to build hierarchical relations between figures and also to make generalisations in novel situations. We cite below some examples which illustrate the use of such models.

Relating squares and rhombuses

Children had made eight different quadrilaterals (figure 2a), some, as they described them, "very very squeezed" and others "very longish" then they compare them.

A: Some are in the middle, some are smaller and others bigger...
L: Anyway, all are different.
T: Which of them will be in the middle?
L: It will be regular.
T: What do you mean by regular?
L: Neither the squeezed ones, nor the longish.

Children, intuitively, seem to recognise a rhombus with equal diagonals as a "regular" quadrilateral. It is apparent from their explanations that, at this point, they do not seem to connect this regular form with the conventional form of a square. Later in this episode, Artemis considers the angle as a way of distinguishing square from rhombus: " rhombus has its sides oblique while the square has them straight".

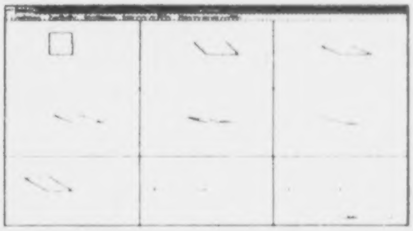

Figure 2a

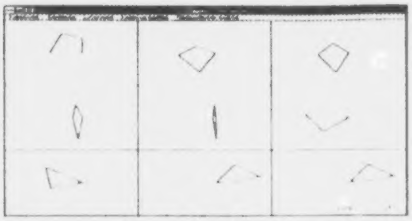

Figure 2b

Relating different classes of polygons

Children had constructed different kites (figure 2b) with the same division of the initial segment. The teacher poses the question: "Can all the four angles be acute?", and the children think that they can. This question comes after they had faced problems such as drawing a kite which was just constructable, and talking about the size of the angles in that case.

Artemis makes a new hypothesis: "In this case one angle must be obtuse because these two sides are small and the other two are large". She comes to the conclusion that she has to have equal sides, and also that in this case it is best to have right angles: "Let's make a square then all the angles are right". She recognises that to have all the angles acute is possible only for the triangle, and she tries to generalise this insight in other types of polygons. She talks about polygons which can have all the angles obtuse and she approximates the circle by suggesting to cut the initial length into 200 segments. All this reasoning is an unexpected behaviour of an 11-year old child considering, especially, her initial visual limitations.

Children's reasoning about the area of quadrilaterals with the same perimeter

Children compare the area of a rectangle with the area of an oblique parallelogram of the same perimeter (episode 7) (figure 3). Lambrini compares the two figures perceptually while Artemis initially extends her previous dynamical models to transform the oblique parallelogram to a rectangle.

Figure 3

L: This will take more. This is more (rectangle)
A: Both are the same.
L: Yes...Both are the same.
T: How do you know this?
L: We know, we saw it.
A: We 've imagined it. If we make straight these two lines (the non horizontal sides of the oblique parallelogram), then this (rectangle) will be the same as that.

In the end of the discussion the teacher expresses her opinion which leads Artemis to find out a way to compare the area of these two figures. She uses her figures to measure the distance between the horizontal sides and then she uses the ruler to measure what she calls the "breadths" of the two shapes. Artemis has intitively developed an appreciation of the role of the height in the area but she cannot use this insight to justify her choice. At the end, she agrees with Lambrini's explanation who uses Artemis's approach to extend her own initial perceptual justifications: " I mean that both have the opposite sides equal, so the sides are equal. But, we' ve made this slant (the rectangle), so it became an oblique parallelogram and its area has become smaller while we were moving the sides". This kind of reasoning shows an integration of perceptual and "dynamical" models that Lambrini used to compare areas by constructing implicitely the "area formula".

Conclusions

Clements & Battista (1992) have underlined the need for research that describes the development of geometric concepts and thinking in various instructional environments. The initial results of our project indicate that, in the specific context of this study, the pupils, starting from visual considerations of the geometrical shapes, had, towards the end of the process, developed connections between the figures and their properties, and formed hierarhical relationships between different classes of shapes. The thinking involved in building these relationships was not just the result of a conjuction of a number of critical attributes that correspond to the figure but it was inseperable from the intuitive and dynamical models developed by the children. Children, by relying on these models, had used reasoning integrating perceptual, imaginary and factual aspects, to justify their choices. We believe that further analysis of our data from this case study, together with the analysis of the work of the remaining groups, will help us to systematise further and extend our observations and interpretations.

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THE QUEST FOR MEANING IN STUDENTS' MATHEMATICAL MODELLING ACTIVITY

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Abstract

The central claim that every mathematical model is based on a certain interpretation of reality motivates the search and examination of the meanings that support students' modelling activity. In this quest for meaning and its evolution we analysed three episodes extracted from a modelling activity developed by a group of four 10th grade students. Our results strongly support the conclusion that students' models are mediated by their particular interpretations and dialogic activity, and by their mathematical, technological and symbolical tools.

Introduction

The mathematical modelling of real world situations is often recommended for its potential in making mathematics meaningful to students. Some of the recurrent questions students have in their minds when they do their mathematics are related to the potential uses, purposes and relevance of the established curricular topics.

Without denying the later conception of meaning, it is our conviction that the problem of meaning is not reducible to the utility or to the relevance of mathematics in solving real problems.

Therefore we have set ourselves the goal of searching for meanings and of sounding their origins in students' modelling activity. We shall do it by closely looking at their interactions and listening to their utterances and by interpreting their actions and words.

Theoretical background: the main stream of our approach

Our approach to the problem of finding meaning in students' modelling activity draws from a set of relevant perspectives which have their stronger references in a sociocultural view of the development of thought and learning.

(1) Learning is viewed as a social and cultural activity. Meanings emerge as a result of a situated and shared activity. The setting where students develop their activity is not a neutral element and their actions are not immune to it.

It is also important to recognise that students' voices are not always unchangeable (Wertsch, 1991). Sometimes they look like interrogative voices; other times they assume a more critical or, on the contrary, a more condescending tone. Students' voices may be more spontaneous or more conditioned and eventually they may sound like the voice of the teacher as in an act of ventriloquism (Wertsch, 1991).

(2) Meanings are mediated by external signs. Some of the important external signs that mediate students meanings are of a linguistic nature (Vygotsky,1987/1993). The

words used by the teacher to describe a real situation that is chosen to be modelled by the class are important mediating elements in the images students make of the situation. The mathematical inputs or clues that are often included in such descriptions can have a pertinacious effect in students' mathematical modelling.

Meanings also permeate students' dialogic activity and their mediating tools. When one of those tools is a technological device, there are additional matters to take into account like the logic of the tool functioning and the logic of its use in a particular task (Skovsmose, 1994).

(3) Students own modelling processes are different from those of experts in crucial aspects. The first mark of distinction between students' modelling and experts' modelling is immediately connected to the setting where the modelling activity takes place. The point is that students are confronted with a problem already formulated in terms of school mathematics — a problem about a real situation or phenomenon which for several reasons is found adequate to be presented to the class. The situation is somehow adapted to a certain group of students and the problems are quite often simplified so that they can stay within students' reach.

The reality that enters the classroom is a *certain* reality, a reality that differs from that of the research laboratory or of the real world exploration conducted by a team of science practitioners (Säljö & Wyndhamn, 1993).

The empirical setting of the study

Our data result from an empirical study developed in a regular mathematics class over a period of four months (Matos & Carreira, 1994). The participants were the students and the teacher of a tenth grade mathematics class.

During the first term the teacher brought simple modelling and application problems to the class and students were organized in small groups to work on them. The teacher undertook the supervision of students' work and she was seen by the groups as a consultant and a guiding resource. By the end of the first term she introduced them to the basic features of the electronic spreadsheet.

In the meantime, she and a team of researchers worked on the elaboration of materials to support a weekly session on modelling problems during the second and third terms. In each of the eleven modelling sessions, three groups of students were observed, videotaped and audiotaped during their activity.

The search for meanings

In our analysis we shall focus on the activity of a group of students. They will be identified as Carla, Sofia and Roberto.

The real situation presented in this class involved the calibration of an hour-glass so that time could be measured in small intervals during the process of running. It was decided to study a hypothetical hour-glass made of two identical conic vessels united by their vertices (see figure 1).

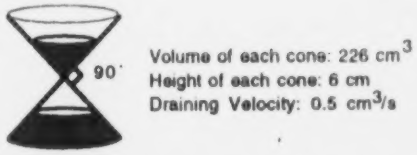
90°
Volume of each cone: 226 cm³
Height of each cone: 6 cm
Draining Velocity: 0.5 cm³/s

Figure 1. The diagram and information provided about the hour-glass functioning

From the entire record of the activity we have selected three episodes found worthy of a close attention.

Episode 1: Local inverses and formal inverses

Sofia initiated the discussion about the hour-glass running. She described it in simple terms, saying that the flowing of the water would cause an increase of liquid in the lower container and a decrease in the upper container. She also mentioned that the water subtracted from the upper cone would be precisely the water added to the lower cone. Carla developed a mental image of the volumes changing, assuming that the upper cone would be initially full. She considered time changing in seconds:

> Carla: "After the first second, it remained 225.5 [in the upper vessel]. That's 0.5 less. After two seconds, it's... 1 less. After three seconds it's 1.5 less, that is, 0.5 times 3. And in the lower one it increased. It became 0.5; 1; 1.5.
>
> Roberto: "The amount that came into the lower one is equal to the amount that came out of the upper one. So, in the upper cone, the volume will be the total volume — which is 226 — minus 0.5 times the time value."
>
> Carla: "And down there, the volume will be the total — which is 0 — plus 0.5 times the time value."

At this point Roberto suggested a way of relating the volumes of the two cones but Carla immediately reacted to his conjecture.

> Roberto: I think that the lower cone must be the inverse of the upper cone".
>
> Carla: "Not the inverse! How could that be, if we're about to see that the two volumes are directly proportional?"

Sofia did not interfere in this dialog and the discussion between Carla and Roberto was not prolonged. The group used the formulae to represent the two volumes as functions of time and graphed simultaneously the two functions (see figure 2).

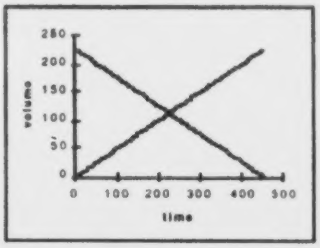

Figure 2. The graph of the two volumes against time obtained on the spreadsheet

As they were interpreting the two straight lines obtained, the notion of inverse came back in a subtle way and, this time, in the comments of Carla:

> Sofia: "This one that goes down is the upper cone."
>
> Carla: "That's right. It starts with... and... it decreases. It looses to the other, and in the other it happens the inverse."

Meanings and their origins

While Sofia was making the description of the hour-glass behavior, she came across several opposite words. She mentioned *increasing* and *decreasing* as well as *adding* and *subtracting* as she spoke about the volume change in each of the cones. Carla began to express this variation in a more quantified manner and Roberto completed her reasoning by introducing new opposites: the water *comes out* of the upper cone and *comes into* the lower cone. The word inversion became a natural sign to represent that very phenomenon. It translated what had been spontaneously captured from the pairs of opposites that filled students' interpretations. We shall say, accordingly, that this type of dialogic activity gave room for the creation of a local model of the hour-glass functioning.

However, when Roberto proposes this same notion to initiate a process of relating both volumes, Carla replies with another voice. She seems to cut out with the local model used so far and she switches to a formal model of inverse. She takes it from a scientific or scholar position and she disagrees with Roberto based on scientific concepts. She stresses her point of view by using mathematical arguments and she shows him that the inverse (the reciprocal) of 225.5 is not equal to 0.5. Everything gets a new formal aspect and the case is closed.

What is important to notice here is that the local model was intercepted by a formal and scientific model. The status of the last one showed to be strong enough to repress the more intuitive and tacit models. The discussion about the relationship between the two volumes did not continue. Nobody questioned the direct proportionality or tried to verify it. No attempt was made to graph the pairs of values in the two volume columns to see what would come up. This is quite in contrast with what is supposed to happen in an expert modelling behaviour since conjectures would be submitted to test and confronted with the real situation.

The conjecture about a direct proportionality only meant a formal argument to reinforce a formal voice within the modelling process. This formal voice turned out to be an important constraint in the modelling process. It held back Roberto's suggestion even though it did not erase the former intuitive model. Obviously, the position of the two straight lines in the graph invigorated the idea of inversion and Carla herself was caught in the temptation of mentioning it.

Episode 2: A graph that is drawn from right to left

Back in the beginning of the session the teacher made some comments to the whole class where she summarize the questions formulated.

> Teacher: In question 1 you' re going to explore how the volume changes with time. In 2, you' ll see how the water level changes with the volume of water. In 3, what is finally required is that you look for the way the water level changes with time. So we may see a transitivity here..."

Once more it was Sofia that tried to figure out how the water level changed as the volume in the upper cone decreased. She draw the following diagram of the situation where she depicted the liquid remaining in the upper cone in successive instants (see figure 3).

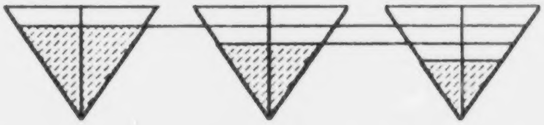

Figure 3. The scheme made by Sofia to represent the level variation in the upper cone

Roberto noted that the water level decreased with the volume of water. Sofia pointed out that constant decreases of level did not correspond to constant decreases of volume.

> Sofia: Only if they were cylinders there would be constant variations of volume for constant variations of level. But as these are cones, the variations are not constant and so it's a different situation."

As soon as students got a formula for the height of the cone as a function of volume, they implemented it in another column of their spreadsheet table. A first reading of the values told them that the level of water started to decrease very slowly.

Roberto took the initiative to draw a sketch on the paper and his representation was accepted by the two girls (see figure 4).

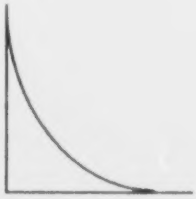

Figure 4. The graph Roberto sketched before plotting level against volume on the spreadsheet

He explained the general idea of his graph where he did not label the axes. Anyway we can see by Roberto's words and gestures that he was referring himself to the labels *level* and *time* on the y-axis and x-axis, respectively.

> Roberto: "The level begins with its maximum value and, as time passes, it decreases until it gets to the minimum value, which is zero, when all the water has come down to the lower container"

When they used the spreadsheet to plot the level against volume they found a different representation from Roberto's (see figure 5).

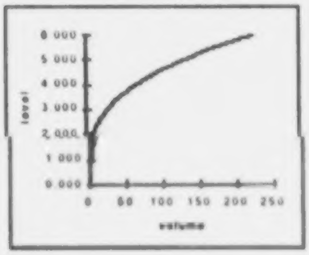

Figure 5. The graph of level against volume obtained on the spreadsheet

At first they didn't understand what was wrong with their initial idea but after a while Roberto came forward to clear up the matter.

> Roberto: "Basically this is the graph that we're expecting with the only difference that it is reverted. Here it's as if time went backwards, as if time was running in the opposite way of ours".

A few moments later, Sofia had a remarkable observation about the graph obtained in the spreadsheet:

> Sofia: "When the curve is drawn here on the computer, it begins from up there (pointing to the right edge of the curve) and it makes all the way to the axis line. So, it's as the starting point was on the right and not on the left. And it really matches with what we were thinking before".

Meanings and their origins

The essential question treated in this episode was to understand how the water level would change with the water volume in the upper cone. In analysing how students dealt with this problem we can see two models of the situation. We will call them the dynamic model and the computer mediated model.

The first model was developed through the pictorial representation made by Sofia. She depicted what she imagined to be the dynamic behavior of the liquid flowing out of the upper cone. In her drawing the time variable was already an important feature of the model. It almost looked like successive shots in a film sequence of the upper vessel. Therefore the conclusion was: the level decreases as the volume of water decreases. But when Roberto proposed a graphical representation, he actually considered the variation of level with *time*. and so students' model grew out of a dynamic vision of the level variation. Even when students obtained the true graph (level against volume) on the spreadsheet, they did not reject their dynamic interpretation. On the contrary, they searched for details that could support such an interpretation. For instance, they claimed that both graphs were compatible if one realised that time was inverted in the computer graph. The fact that the computer plotted the graph curve from right to left was used as a good argument to sustain the idea of time being inverted.

What has been described suggests the conclusion that the time variable became very central in students' reasoning. Although having accepted the computer graph, they used a feature of the computer output (the graph is drawn from right to left) to reinforce their own perception of the relation level-time.

Episode 3: Why is a cone different from a cylinder?

Students finally plotted the graph level against time on the spreadsheet to get a better insight of how time should be marked on the hour-glass (see figure 6).

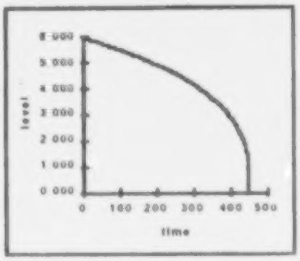

Figure 6. The graph of level against time obtained on the spreadsheet

They came to realise that Roberto's previous graph indicated a slow decreasing towards the end of the emptying which did not match the real phenomenon. This way students revised their graphical representation of the function level-time. The new graph provoked some discussion between the students. They analysed the type of variation represented and discussed it.

> Carla: "When there isn't much water left and the cone is almost empty, then it goes down faster, doesn't it"?
>
> Sofia: "Yes, and the graph translates the water movement as it goes down in the upper cone".
>
> Carla: "That's right. At start, when there's still a lot of water, it has to be decided which is going down first... (smiling). But when there's only a little liquid, then it all goes down at once".
>
> Sofia: "We got this curve because we're talking about a cone. If it was a cylinder it would be a linear model".
>
> Roberto: "True, for in that case the water flow would always be the same... The width of the container would not change. But if it's a cone, and if it's quite full... the water flow is different. It makes a difference in the water flow if the cone is quite full or if it's almost empty".

Meanings and their origins

Two ideas seem to cohabit in students' reasoning. One is the idea that the shape of the vessel makes the level variation non-linear. The other is related with the notion of an increasing speed of water flow as the cone gets emptier.

Carla used a metaphor where she insinuates that the liquid flow would not have a constant speed. Only Sofia kept an accurate idea about the influence of the vessel shape in the level function. Roberto aimed at combining both perspectives. In a way he followed Sofia's interpretation when he noticed that there was a variation in the width of the cone. But at the same time he went along with Carla's image according to what the water did not flow at a constant rate and in the beginning it would be slower than in the end. It all seems to indicate that Roberto tried to accommodate both views of the problem.

What happened then were three voices raising to explain the non-linearity. Sofia's voice pointed to the shape of the vessel, Carla's to the changeable speed of water flow and Roberto's showed a mixture of both. All of them brought different meanings to the situation but those meanings were not seen as conflicting. On the contrary, they were combined in a two-sided version of the phenomenon.

Conclusions

To look for meaning in students' modelling activity implies the recognition that students' modelling behavior is fundamentally tied to the setting where the modelling takes place. There are all sorts of mediating elements contributing to the emergence of meaning. We have identified and discussed some of those elements, namely students' dialogic activity, the role and value of everyday language, its contextualization in the analysis of the situation, as well as the uprising of formal voices, sometimes in a ventriloquism of the voice of reason.

We have also looked at the way some of students' models come from their intuitive views of the problem and how they can become rather persistent even when in the presence of contradictions. Students can find their own processes of accommodating different interpretations and concepts and of using their tools to reinforce such processes.

In many aspects there are notable differences between what students do in their mathematical modelling activity and what is supposed to be the performance of a modelling expert. This will not come as a surprise if one is willing to appreciate the reasons why students come up with their own senses of a certain real situation and its mathematical grounds.

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THE ROLE OF IMAGERY AND DISCOURSE IN SUPPORTING THE DEVELOPMENT OF MATHEMATICAL MEANING

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The study reported in this paper investigates the role that situation-specific imagery and classroom discourse play in grounding students' activity as they develop mathematical meanings. Issues addressed in this paper emerged during a teaching experiment which was conducted in one first-grade classroom. The teacher in the study participated as a collaborating member of the research team. The notion of imagery as discussed in this paper is related to Pirie and Kieren's (1989) recursive theory of mathematical understanding. The analysis presented provides an example of the general issue of maintaining a grounding in imagery and its potential importance in teacher development efforts.

This paper focuses on the role that situation-specific imagery and classroom discourse play in grounding students' activity as they develop mathematical meanings. The issues addressed in this paper are related to Pirie and Kieren's (1989) recursive theory of mathematical understanding. They argue that mathematical understanding is a "recursive phenomenon and recursion is seen to occur when thinking moves between levels of sophistication" (Pirie and Kieren, 1989, p. 8). The first level they discuss is characterized as "primitive doing." Mathematical actions at this level typically involve the use of physical objects, figures, or symbols. The first recursion occurs when the learner is able to form images out of this "doing" and is called "image making." At the next level, the images are replaced by a form for the image. This "image having" no longer requires particular actions as examples and is considered a first level of abstraction. Pirie and Kieren (1989) note that "it is the learner who makes this abstraction by recursively building on images based in action" (p. 8). As the learner is able to examine the images for specific properties, the next level occurs. It is important to note that for Pirie and Kieren (1989), the actual process of learning does not proceed in a linear manner through the levels. While activity at an inner level can be used to build more complex ways of knowing, it is often necessary for the learner to *fold back* to a previous level. When one does *fold back* to a previous level, the action at that prior level is not the same. The experiences at the outer level serve to inform the re-visited inner level. It is this recursive process that makes mathematical understanding a dynamic process. It is important to note, however, that Pirie and Kieren's notion of *folding back* was developed as a way to describe shifts in individual activity. As the focus in this paper is on classroom discourse, we will adapt Pirie and Kieren's notion to describe shifts in collective activity.

The sample episode discussed in this paper is taken from a first-grade classroom in which the teacher participated as a collaborating member of a research and development team. Our intent is not to offer an example of exemplary teaching, but to illustrate the importance of imagery and discourse in supporting the development of mathematical meaning. This paper will be of more than local interest if it serves as a paradigmatic case that can both help others develop understandings of their own practice and contribute to the growing research literature on effective teaching.

In the following sections of this paper, we will first describe the teacher and classroom that are the focus of this paper, and then outline the data corpus. Against this background, we will present a classroom episode intended to exemplify the importance of students' activity remaining grounded in situation-specific imagery. We conclude by discussing the significance of imagery-in-discourse for the development of mathematical meaning.

Ms. Smith's Classroom

The majority of the eleven girls and seven boys in Ms. Smith's first-grade (age six) classroom were from middle or upper middle class American backgrounds. There were no minority children in the classroom, although a small percentage attended the school. The students in the class were representative of the school's general student population. Although not a parochial school, morals and values were considered to be part of the responsibility of schooling and children regularly participated in religious activities.

Ms. Smith's classroom is of particular interest because an analysis of videorecorded interviews conducted at the beginning and end of the school year indicates that the students' conceptual development in mathematics was substantial. Students who, at the beginning of the year, did not have a way to begin to solve the most elementary kinds of story problems posed with numbers of five or less had, by the end of the year, developed relatively sophisticated mental computation strategies for solving a wide range of problems posed with two-digit numbers.

The teacher, Ms. Smith, was a highly motivated and very dedicated teacher in her fourth year in the classroom. She had attempted to reform her practice prior to our collaboration and voiced frustration with traditional American mathematics textbooks. Although she valued students' ability to communicate, explain, and justify, she indicated that she had previously found it difficult to enact an instructional approach that both met her students' needs and enabled her to achieve her own pedagogical agenda. When we began working with Ms. Smith, it soon became apparent that she continually reflected on and assessed both the instructional activities she used and her own practice. In addition, she had a relatively deep understanding of both mathematics and her students' thinking. Ms. Smith was seeking guidance with her reform efforts; we were seeking a teacher with whom to collaborate as we developed sequences of instructional activities.

Data Corpus

Data were collected during the 1993-94 school year and consist of daily videotape recordings of 103 mathematics lessons from two cameras. During whole class discussions, one camera focused primarily on the teacher and on children who came to the whiteboard to explain their thinking. The second camera focused on the students as they engaged in discussions while sitting on the floor facing the whiteboard. Additional documentation consists of copies of all the children's written work; daily field notes that summarize classroom events; notes from daily debriefing sessions held with Ms. Smith; and videotaped clinical interviews conducted with each student in September, December, January, and May.

A method described by Cobb and Whitenack (in press) for conducting longitudinal analyses of videotape sessions guided the analyses. This method fits with Glaser and Strauss' (1967) constant comparative methods for conducting ethnographic studies. It involves constantly comparing data as they are analyzed with conjectures and speculations generated thus far in the data analysis. As issues arise while viewing classroom videorecordings, they are documented and clarified through a process of conjecture and refutation.

In the following section, we will present an episode which highlights the importance of imagery and discourse in ensuring that students' activity remains grounded. The episode provides an example *folding back*.

Classroom Episode

Our collaboration with Ms. Smith between March and May of the school year focused on the development of an instructional sequence called the empty number line. The empty number line is a number line which is "empty" of or lacking in numerical increments. This sequence was designed to support the development of counting-based concepts of ten, and strategies for estimation and mental computation with two-digit numbers. The initial instructional activities in the sequence as it was originally outlined by its developers involved the use of a bead string composed of 100 beads (Treffers, 1991). The beads were of two colors and were arranged in groups of ten. We decided to modify the sequence by omitting these instructional activities because the bead string did not serve as a means by which children might explicitly model their prior problem solving activity. Instead, we attempted to develop the empty number line sequence by building on the scenario of a candy shop. Students would be given amounts of candies in the candy shop and asked how many candies the candy shop owner would have if she either made more candies or sold candies. The empty number line was used to record these transactions where incrementing or decrementing on the number line corresponded with the appropriate candy shop activity (see figure 1).

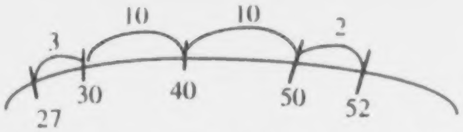

Figure 1. Twenty-seven candies and she makes 25 more.

As the sequence progressed, we inferred that most of the students' activity was grounded in situation-specific imagery of activity in the candy shop. One of the primary sources of evidence was the fact that students, working in pairs, created empty number lines to tell stories about what happened in the candy shop. However, a classroom episode that occurred on March 30 provided convincing evidence that, for at least some of the students, acting with the empty number line had become a purely calculational activity that was situated in purely calculational discourse involving explanations speaking "exclusively in the language of *number and numerical expressions*" (Thompson, Philip, Thompson, & Boyd, 1994, p. 8, italics in original).

In this crucial lesson, Ms. Smith posed addition and subtraction tasks by drawing a horizontal empty number line, and by describing transactions in the candy shop. One of the tasks posed (see figure 2) was *Mrs. Wright has 90 pieces of candy and she sells eight of them. How many candies does she have left?*

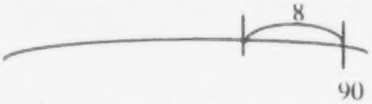

Figure 2. Ninety candies in the shop and she sells eight.

In the subsequent discussion, Bob explained the following solution :

Bob: I think it's eighty-one. Because if we're already down to ninety and then you don't count. . . you don't count ninety because if you have eighty and then you take away nine, it would get you down to eighty.

Here, Bob argued that the entity signified by "ninety" should not be included in those taken away. As a justification, he observed the "90" would not be counted when counting backwards. On this basis, he reasoned that as the result of taking away nine would be 80 (i.e., 89, 88, . . . , 81), the result of taking away eight is 81. A detailed analysis of this and other contributions to the exchange revealed that while most of the children gave number words and numerals quantitative significance, there were qualitative differences in their individual interpretations of terms such as "ninety." This in turn indicated the absence of taken-as-shared imagery underlying the number line (see Cobb, Gravemeijer, Yackel, McClain, &

Whitenack, in press). A range of interpretations emerged, as evidenced by Dan's explanation of his answer of eighty-three.

Dan: I think it's eighty-three because I'm counting the ninety as a number.
T: Now we're talking about pieces of candy. There're ninety pieces of candy. Okay?
Dan: Okay, if you have ninety pieces of candy and. . . and it couldn't be eighty-two 'cause you'd be. . . 'cause the ninety, if you're taking away it and counting one, well it would just be the ninety-eight [sic]. It. . . you would be counting something extra. So you would take away two. . . the ninety-eight and the ninety-nine [sic].

Dan appeared to misspeak when he said ninety-eight and ninety-nine instead of eighty-eight and eighty-nine. At first glance, it might seem that he had arrived at his answer of eighty-three by counting backwards starting from ninety rather than eighty-nine. However, his comments in the remainder of the episode indicated that his interpretation of the task was relatively sophisticated. Collectively, these comments suggest that, for him, the ninth decade when counting comprised 80, 81, 82, . . . , 89. The "ninety" to which he referred appeared to be of special significance in that it signified an additional item beyond this decade. By this reasoning, the solution to 90 - 8 involved taking away "the ninety" and then seven from the decade 80, 81, 82, . . . , 89. The result for Dan was then eight-three rather than eighty-two.

T: We're at ninety. . . we have ninety pieces. You said if you took away one of those pieces you would have. . . (she notates, see figure 3).
Dan: Eighty-nine. If you took away another one you would have eighty-eight (Ms. Smith notates, see figure 3). Now, you've got three pieces away. Now, you take away. . .
T: Now wait a second. You took away, Dan, you said you took away one piece, and that left you with eighty-nine. Is that what you said? (Dan nods in agreement). Then you took away one more piece, and that left you with eighty-eight. Is that right? (He nods again.) So how many pieces have you taken away so far?
Dan: Three.
T: Okay, show me where are the three pieces you took away.
Dan: The ninety, ninety-eight, I mean the ninety, the eighty-nine, and the eighty-eight.

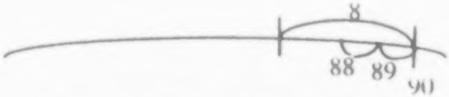

Figure 3. Dan's solution to the task.

For Dan, three rather than two candies had to be taken away to leave eighty-eight because there was an additional candy beyond the ninth decade, "the ninety."

Following this line of reasoning, ninety take away two would be eighty-nine because it was necessary to take away the ninety and then one from the ninth decade. Although it could be argued that the numbers held quantitative significance for Dan, the lack of a taken-as-shared interpretation of the number line led to Dan and Ms. Smith talking past each other. Ms. Smith's recasting of questions in terms of candies was insufficient because there was a lack of shared interpretation of the number line itself, not of the original task situation that involved candies.

This lack of basis for communication continued to be evidenced as several students disagreed with Dan's claim that eighty-three candies would be left. To this Dan responded:

Dan: I still think it's eighty-three 'cause if you're counting the ninety which you have to 'cause if you have ninety pieces and if you didn't count ninety, you'd just have eighty-nine.

Dan's clarification of his position led to a discussion about whether or not to count "the ninety." At this point, it appeared that students were unable to effectively communicate. Their explanations appeared to carry the significance of acting on experientially-real arithmetical objects. The difficulty arose from the fact that Ms. Smith and the children interpreted the empty number line in a variety of different yet personally-meaningful ways. They were, in fact, unable to establish an adequate basis for communication during the remaining ten minutes of the lesson. This remained true even though Ms. Smith redescribed several of the students' explanations in considerable detail by referring to the empty number line she had drawn. In the absence of taken-as-shared imagery for the empty number line, the situation remained unreconcilable.

In reflecting back on the lesson, Ms. Smith stated that she believed that the students' explanations were grounded in the imagery of the situation. She based this judgment on the fact that students were able to talk explicitly in terms of the candies while explaining and justifying their solutions. This interpretation caused her to judge that it was unnecessary to initiate a *folding back* process by, say, introducing unifix cubes as substitutes for candies. Although we agreed that most of the students' explanations carried quantitative significance, we agreed that they were attempting to give the empty number line meaning in terms of re-presented counting activity. Difficulties in communicating arose because there was nothing taken-as-shared beyond the empty number line that the students could point to to explain their counting-based *interpretations.*

As a consequence of these discussions, Ms. Smith introduced unifix cubes in conjunction with the empty number line. Our rationale was that the cubes might be constituted as countable items. With Ms. Smith's guidance, the children might *fold back* by looking at the cubes though counting and acting with the empty number line. The children had previously made bars of ten unifix cubes to model packing candies into rolls of ten. Ms. Smith capitalized on this prior activity by arranging the unifix cubes in bars of ten of differing colors. Laying the bars end to end in the tray under the white board, Ms. Smith drew the empty number line above the cubes. She re-introduced the cubes as candies and cast all problems in terms of the candies. The first task posed involved incrementing and decrementing. When students offered their solutions, she recorded them on the empty number line and partitioned the train of unifix cubes at the appropriate points to correspond with the jumps on the empty number line positioned directly above the cubes (see figure 4).

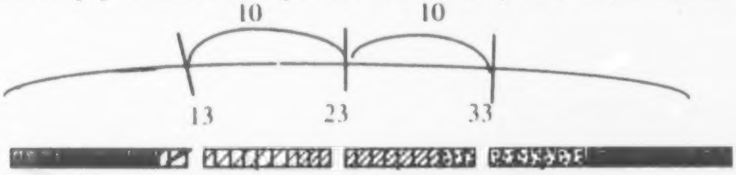

Figure 4. Supporting incrementing activity with the cube train.

Further, she on occasion, specifically asked the children to determine which pieces of candy would be needed to get from, say, thirty-four to forty.

Several students appeared to modify their interpretations of the empty number line as they participated in discussions that made reference to the cubes. For example, Dan explained his solution to the task *Ms. Wright has twenty candies in the shop and she sells two* by stating:

Dan: You take away the twenty and the nineteen leaves eighteen.

In general, the development of a taken-as-shared basis for communication involved a recursion such that acting on the empty number line and counting were renegotiated as the children looked at the train of cubes through these activities. By initiating a *folding back* of discourse, Ms. Smith made it possible for her students to develop taken-as-shared imagery for the empty number line. In Pirie and Keiren's terms, they might be said to "have" an image of the train of cubes as informed by outer level knowing so that they could subsequently act independently of it, but in ways such that they made taken-as-shared interpretations.

Conclusion

Throughout this paper, we have attempted to illustrate the importance of imagery and discourse in supporting the development of mathematical meaning. While Pirie and Kieren (1989) argue that the imagery of the situation emerges from initial "doing" which ultimately supports students' construction of mathematical conceptions, this is a recursive process that requires students revisiting prior levels of image doing, making, and having. As a result, the overarching imagery of the context as negotiated in classroom discourse is crucial for the development of mathematical understanding to emerge.

In Ms. Smith's classroom, the imagery of the situational context was highly valued. Although she typically engaged her students in skillfully-developed narratives, the possibility of initiating a *folding back* of discourse did not occur to her spontaneously. This observation is extremely significant in our view. Elsewhere, we have documented that she was, in many ways, an unusually gifted and reflective teacher (see McClain and Cobb, 1995). The fact that she rarely initiated the *folding back* of discourse leads us to conjecture that few other teachers initiate such shifts in discourse. This in turn suggests that these shifts and the general issue of maintaining a grounding in imagery be given particular attention in teacher development efforts.

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THE ORIGINS AND DEVELOPMENT OF THE NCTM *PROFESSIONAL STANDARDS FOR TEACHING MATHEMATICS*

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The reform of mathematics education in the USA has been led by the NCTM Standards for curriculum, teaching, and assessment. This paper reports part of a case study of the NCTM Standards project considered as an effort toward systemic change. The story of the Teaching Standards provides an interesting example of creativity in writing recommendations for teaching, even though the Teaching Standards have received less attention than the Curriculum and Evaluation Standards, probably because of differences in funding, organization of the writing teams, dissemination, and timing.

In 1991 the National Council of Teachers of Mathematics (NCTM) published the second of its standards documents, the *Professional Standards for Teaching Mathematics* (NCTM, 1991). Although this document, usually referred to as the *Teaching Standards*, has been well received in mathematics education, it is not as well known as the first of NCTM's standards documents, the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). This paper reports some of the reasons for the differences in the development and impact of the two documents. The analysis is part of a case study of the first two NCTM standards documents (McLeod, Stake, Schappelle, Mellissinos, & Gierl, in preparation).

The purpose of our case study was to understand the origins, development, dissemination, and impact of the NCTM *Standards* as an example of a systemic change effort. Our methods followed the recommendations of Stake (1994). Main sources of data included interviews with NCTM leaders and state mathematics supervisors in the US. One of eight studies of educational change in the US (Romberg & Webb, 1993), our project is part of an international effort coordinated by the Organisation for Economic Cooperation and Development.

Origins of the *Teaching Standards*

The initial plan for the NCTM *Standards* came out of the work of the NCTM Instructional Issues Advisory Committee. The plan included the preparation of standards for curriculum, instruction, and evaluation, all of which would have been included in one document. Although some see the origins of this plan in the publication of *A Nation at Risk* (National Commission on Excellence in Education, 1983), others say the plan arose out of concerns from NCTM members that recommendations on mathematics textbooks were too often subject to political and other influences from outside of the professional mathematics education community (McLeod, Stake, Schappelle, & Mellissinos, 1995).

As the NCTM Board of Directors planned the development of the standards project, the *Teaching Standards* were delayed, even though many people thought that teaching should come first because of its importance. As one leader described the conflict, "I must admit that my own predilection was to start with teaching, because I thought the classroom and what teachers do with kids is probably at the center. But that isn't what NCTM wanted." NCTM President John Dossey recalled how the NCTM Board decided to focus first on the *Curriculum and Evaluation Standards:*

> We were talking about doing both curriculum and evaluation as well as the teaching standards in one document. We realized that it would be too big of a change. We felt that it would be a difficult enough task just to get everybody to say, "We want to try and change our content and our view of content."

Since NCTM had not been able to find any funding agency that was willing to support the entire standards project, separating out and delaying the development of standards for teaching made the initial project smaller and easier to fund. In some ways, the decision to focus on content first was just "being practical," as one leader put it. Leaders and writers generally agree now that it was the right decision. As one state supervisor put it:

> It was absolutely crucial to have the *Curriculum Standards* be the banner. Very few people really remember that there are also evaluation standards in the document; they sank without a trace. If there'd been a comprehensive document called *Curriculum, Teaching, and Assessment Standards,* teaching and assessment would have dropped off. What would have gotten the attention was the content standards.

NCTM leaders, however, were also aware that it was important to work on changing all parts of the educational system. The experience of the 1960s suggested that you need to "do it all," in the words of one leader. For example, NCTM was very aware that the "widespread shortage of qualified mathematics teachers" (NCTM, 1980, p. 24) was a continuing problem, and any move away from traditional instruction would require strong programs for staff development as well as teacher preparation. However, the final decision to begin with content and curriculum received broad support.

Comparing the Two Standards Documents

There are many similarities between the development of the *Professional Standards for Teaching Mathematics* (NCTM, 1991) and the *Curriculum and Evaluation Standards* (NCTM, 1989). Each document began with the appointment of a Task Force by the NCTM Board of Directors, the preparation of a proposal by the Task Force, and the approval of funding for the project from the NCTM Board of Directors. Although NCTM had to provide almost all the funding for the *Curriculum and Evaluation Standards* (NCTM, 1989), the situation had changed by 1989. As one NCTM leader put it, now the National Science Foundation (NSF) "was almost inviting us to come for money," and a grant proposal requesting funding for half the costs was submitted to NSF.

The proposal to NSF outlined the major tasks. Writing teams were to prepare a draft document in the summer of 1989, and NCTM would hold hearings and gather feedback during 1989-90, a year of discussion and review. Finally, a revised draft would be prepared during the summer of 1990, based on feedback from the field, and NCTM would produce the final document in 1991. The plan included standards for teaching (i.e., what a mathematics teacher should be able to do), standards for the professional development of teachers, and standards for the evaluation of teaching. These three areas appeared in the final version of over 200 pages (NCTM, 1991), along with a fourth section on standards for administrators and policymakers on support for mathematics teachers.

The proposal listed a commission (chaired by Glenda Lappan) and three working groups (led by Deborah Ball, Susan Friel, and Tom Cooney). Each working group had an "assistant/reactor" who had special responsibility for responding to what the leader wrote. In contrast to the 1989 *Standards*, when writing teams met in 1987-88 for three periods of two weeks each to discuss and debate the ideas, the plans for the *Teaching Standards* included three-day meetings of the working groups. There was less time (and less funding) for group meetings, and leaders were assigned more of the responsibility for the writing. The entire working group met to plan the writing and later to respond to the lead writer's drafts. Individual members of the groups still made significant contributions, but there was substantially less time for group interaction than in the preparation of the *Curriculum and Evaluation Standards*.

This change was decided upon in part because of the need to reduce costs; it was less expensive to assign the main part of the writing to just the leaders. But the decision was also a consequence of the difficulties with the quality of the writing in the drafts of the *Curriculum and Evaluation Standards* (NCTM, 1989).

The changes in responsibility for writing, along with the reduced amount of time that the writers spent together, are probably the main reasons for the different impressions one gains from talking to members of the working groups for the two documents. The esprit de corps in the writing groups for the *Curriculum and Evaluation Standards* was especially strong; in the case of the *Teaching Standards*, the sense of group ownership seemed somewhat less intense. As one leader noted, "A certain camaraderie developed in the *Curriculum and Evaluation Standards* that was missing in the *Teaching Standards.*"

The change in the structure of the writing groups appeared to be the main source of the change, but the difference in leadership style (noted by several participants) may have been a contributor as well. As a leader put it, the two leaders of the 1989 and 1991 projects (Tom Romberg and Glenda Lappan) "had different leadership styles--Lappan talked about that in the first meeting. She was not going to be a hands-off leader." Romberg was often willing to remain above the fray of the writers' debates and rarely got involved in arguments. Lappan, on the other hand, was seen as a leader who provided "direction without being dictatorial." Both Romberg and Lappan received many positive comments from the writers who worked with them.

Changes in funding, in leadership and in leadership style, in writing assignments, and of course in the topic under discussion (pedagogy rather than content) contributed to making the *Teaching Standards* different from the 1989 *Standards*. Perhaps the most significant factor of them all is just that the *Curriculum and Evaluation Standards* came first and made a big impression, leaving the *Teaching Standards* always in the "bow wave" created by the arrival of the earlier document.

The Writing of the *Teaching Standards*

The writing groups met in the summer of 1989 and began their work. Since only Lappan and Cooney had been involved in writing the 1989 *Curriculum and Evaluation Standards*, some of the same issues about the meaning of the term *standards* came up again. Some writers wanted to specify standards that would be criteria to judge quality, but most writers (like the writers of the 1989 *Standards*) had by this time moved away from an accountability perspective of standards

(McLeod et al., 1995). More of the new writers were concerned that teachers would find the idea of standards to be too prescriptive, even arrogant, and they worried about how the phrase "professional teaching standards" would be interpreted by teachers. The discussions were intense, and brainstorming sessions encouraged consideration of many different ideas about standards.

One concern was that the teacher unions in large cities would immediately reject the notion of standards. This concern was part of the reason that members of Deborah Ball's writing group decided that they wanted to express the spirit of teaching--an idea that was consistent with telling stories of teaching, rather than stating standards or specifications. The stories of teaching were called *vignettes*. The use of vignettes to describe teaching led in an interesting way to the development of standards. Deborah Ball recalled:

> We met for a day and we were thinking about what the different vignettes ought to be like. The people in my group thought that it seemed reasonable that we would write vignettes, and we talked a bit about where we would get them and what a vignette would be. At that point, we were imagining the text as being vignettes. In trying to conceptualize how we could have an array of vignettes that would capture differences in teaching for understanding, we backed into having to talk about what dimensions of teaching would be like, or what you would have to think about so that you could have variation among our vignettes. Over night from one day to the next, I remember saying at the end of the day, "Maybe I could try to write down some of the dimensions that seemed to be emerging across the day as we were talking."

The dimensions that came out of the vignettes were the basis for the opening section of the *Teaching Standards* (Standards for Teaching Mathematics), with the first six standards on mathematical tasks, discourse (the teacher's role, the students' role, and tools for enhancing discourse), learning environment, and analysis of teaching. The focus on discourse was controversial at first, but eventually the term came to be one of the hallmarks of the *Teaching Standards*. Initially the resistance to discourse came from the reluctance to introduce a term that many teachers would find foreign. Writer Tom Schroeder recalled:

> It was Deborah Ball who proposed using the term *discourse*, and I believe that she was familiar with a body of research that uses that term in the sense laid out in the *Teaching Standards* (NCTM, 1991, p. 34). There were some discussions in our working group on the pros and cons of the term, including some wisecracks about "discourse, dat course, and dee udder course."

A state mathematics supervisor was particularly critical in an early review, but eventually was won over, as Deborah Ball reported:

> At first he absolutely hated what we had done. He was one of those negative reviewers. He wrote us this angry, vehement letter about what a stupid word *discourse* was--academic jargon. He disagreed with the framework completely. He went right to the jugular on the whole framework that we did. That was the first summer. We thought a lot about his comments; I didn't completely agree with him, but I really thought hard. I thought hardest about the responses that were vehement because they were the most useful. When we did the revision, which still kept discourse in a more prominent place, he fell in love with them. In his second review, he wrote "not only is this an unbelievable document, but it's scary." What he meant by that was it's scary because it has such a radical vision of teaching. He thought that if people really got it the way that he was interpreting it, they'd be frightened. Now that's one read. But I think that's a good read.

When that state supervisor was asked about how much teachers knew about the two *Standards* documents, he expressed concern about the lack of attention to the dissemination of the *Teaching Standards:*

> I think that the *Professional Teaching Standards* are harder to attain, but more significant by far than the *Curriculum and Evaluation Standards.* I'm telling my friends at NCTM now that we've got to add Addenda programs and videotapes and all those kinds of things to help capture what we mean by tasks and discourse and environment.

The Dissemination of the *Teaching Standards*

The publication of the *Teaching Standards* in 1991 was a major event for NCTM, and the dissemination plans were similar to what was done for the *Curriculum and Evaluation Standards* (NCTM, 1989). The document was distributed free to 58,000 members and sold (for $25) to 63,000 others, making it one of NCTM's best sellers, even though it had less than half as many copies distributed (by mid-1995) as the *Curriculum and Evaluation Standards.* The Executive Summary was also circulated widely, just as for the 1989 *Standards.* NCTM meetings and journals now focused on the *Teaching Standards,* as well as the *Curriculum and Evaluation Standards.* Nevertheless, the dissemination effort did not seem to have the same impact as the 1989 *Standards.* As one state supervisor put it:

> I do not believe that there is anywhere near the level of awareness about the *Teaching Standards* as about the *Curriculum and Evaluation Standards*. The *Teaching Standards* have not been as well disseminated, not been as widely discussed, and have not received the attention that they require. We've not had as many meetings on them, and they have gotten lost in the shadow of the *Curriculum Standards*.

Most leaders do not think the *Teaching Standards* have been lost in the shadows, but many have deep regret that part of the plans to disseminate the *Teaching Standards* were never realized. The *Curriculum and Evaluation Standards* had been disseminated in every state through the "Leading Mathematics Education into the 21st Century" project of the Association of State Supervisors of Mathematics, but the corresponding project for the *Teaching Standards* was never completed, to the great disappointment of a number of state supervisors of mathematics, who felt that the 21st Century project had been extremely successful. As one state supervisor noted:

> We decided that it shouldn't be the state supervisors dealing with the dissemination of the *Professional Teaching Standards;* it really needed to be the National Council of Supervisors of Mathematics. The reasoning was that curriculum is more of a state responsibility, and instruction is more for the districts [and the local supervisors]. It really made sense. [Unfortunately, the project] got lost in the shuffle; it fell through the cracks.

The bitterness of some state supervisors was strongly felt, and some preferred not to talk about the circumstances. One NCTM leader noted that all of the work of preparing the proposal was essentially done, but there was just a failure of leadership. Although the dissemination of the *Teaching Standards* was not carried out to the extent planned, the document was seen by many leaders as an innovative and effective contributor to the reform effort in mathematics education. Its use of language has been identified as a particular strength. Tom Schroeder noted how terms like *discourse* were used in special ways in the document, an example of what he called "*Standards*-speak":

> Each of the *Standards* documents uses terms (some might say jargon, but I won't use that pejorative) that are a bit unfamiliar or unusual, in order to draw attention to issues and re-frame them more broadly than might otherwise be the case. The terms I have in mind include "mathematical disposition"

instead of "attitudes toward mathematics;" and "discourse" as opposed to, say, "classroom communication." The term "vignette" is another instance of *Standards*-speak with a purpose.

Although the *Teaching Standards* are admired by many, there have been criticisms of the document. Some in the mathematics community have been concerned that mathematicians did not have sufficient influence on the document; they may not know that many of the writers were professors of mathematics. A criticism from an NCTM leader raised a more substantive issue about the kinds of support that teachers need to build curriculum:

> One of my problems with the *Teaching Standards* is that it doesn't talk about how to put a collection of tasks together to make a cohesive unit, a cohesive year, and a cohesive curriculum across years. You have to focus on more than the tasks that kids do.

Criticism of the standards movement has grown in recent years, and there are indications that federal support for the movement may be coming to a halt. Nevertheless, the *Teaching Standards* (NCTM, 1991) were judged by most NCTM leaders as a very successful project during NCTM's finest hour.

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MATHEMATICS AND THE SIGN

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The *20th century has seen a growing recognition that language provides the framework through which we develop our understanding of the world. This paper examines the basic notions of Ferdinand de Saussure (1857–1913), a preeminent figure in the development of linguistics and the foundation of structuralism, whose revolutionary work set the scene for an innovative perspective on reality. It is a model of a reality constructed through linguistic signs by the individual who is in turn constituted by his social environment and the interpersonal systems of norms he assimilates as culture. Language, it suggests, is the medium through which, and in which, mathematical ideas are formed and exchanged.*

Introduction

Speaking of the insights and issues which Saussure's contribution to linguistics posed, Walkerdine remarks: "Saussure was writing seventy years ago, but we may presume that in our time none of these questions has been satisfactorily answered. Yet the questions, in essence, remain and may be fruitful for developmental psychology. For example, how do children come to read the myriad of arbitrary signifiers - the words, gestures, objects, etc. - with which they are surrounded, such that their arbitrariness is banished and they appear to have that meaning which is conventional?"(1988 p.3). Theories of hermeneutics, phenomenology and post-structuralism (eg. Brown 1994a, 1994b, in press) go some way towards addressing issues involving understanding and language. Theories of constructivism which are prominent in mathematics education discourse at present, even those based in dialogue (eg. Ernest 1994), could perhaps develop their potential to address such questions. Von Glasersfeld (1995 pp. 129-45) introduces Saussure as a linguistic model to augment his account of the semantic basis of radical constructivism.

The Sign

Saussure provided many insights into issues involving understanding and language through his analysis of *la langue*: a language system of words combined with a set of rules, values and norms. He identified within it the two states of synchrony and diachrony, states of linguistic stability and change, respectively. Synchronic analysis became the study of the structure of language during stable phases within geographic areas. Diachrony, with its binary perspectives, traced the prospective and retrospective evolution of linguistics; knowledge of which should not, in his opinion, be allowed to affect the sign's relationship with other signs. Saussure portrayed *la langue* as a social institution, but endowed each person with an internal representation of it; thus permitting it access to *la parole*, its realisation in every day acts of speech and writing (Saussure 1983 pp. 23-35 & pp. 114-17).

Saussure's model of the sign is entirely cognitive; it is that of a two sided psychological entity which comprises the signifier, or sound image, and the signified, or associated concept or meaning. The sign constitutes the basic linguistic unit and, significantly, both elements of it are mental phenomena; not to be confused with either the physical sounds relating to the signifier, nor the actual referent relating to the signified. This clear stance is adopted to deliberately exclude all contact with the non-linguistic referent which some critics have claimed is a major flaw in the theory (cf Walkerdine 1982). Saussure's argument would, I think, be that the referent is, in every way, external to the study of *la langue*: what matters are the internal psychological structures and relationships. Saussure identified two basic principles of the sign, the first being that the relationship between the signifier and signified was arbitrary, and the second, being the linear nature of the signifier. The latter provides a point of contact with both individual speech acts, and the real time dependent world; it leads to the observation that signs cannot signify in isolation but only within a linguistic system. Clearly to produce particular meanings the signs must be combined together in a certain way, but it is the linguistic system of values which gives a word its form, not its material manifestation (Saussure 1983 pp. 97-103).

Saussure suggested that the value of a sign was produced by its relationships with other signs in one of two ways. Syntagmatic relations exist with signs that surrounded it in syntagmas, for example, the meaning of the word 'green' is deferred until it is combined in an act of *parole* with other words: Sarah Green, green grass, Greenpeace. Associative relations exist between signs in mnemonic groups: psychological associations with words that could be used to replace, contrast with, or combine with, the sign. If I am angry, for example, there are a plethora of terms which I can use to describe my feelings (hurt, irritated, annoyed); each one limits the range of applicability of the others (ibid pp. 170-75). Saussure's distinction between these two types of relation is however far from well defined: there are many examples of psychological associations which derive from frequent occurrence in familiar phrases.

Saussure's central tenet was perhaps that, "... in language there are only differences, and no positive terms" (ibid p. 166). The writing system exemplifies this: its only requirement is that the letters and words which we write should look sufficiently different from others with which they may be confused. Saussure also explored the identity of signs; he differentiated between 'material' and 'linguistic' similarity, perceiving the latter to be one of form not substance. To illustrate this he gave the example of the '8.25 p.m. Geneva to Paris Express': he observed that we refer to it as if it were a particular train, the same every day. Inevitably, however, both the rolling stock and the personnel will be different; hence he concluded, the '8.25 p.m.' was simply a construct, useful to differentiate it from other trains, but not physically manifest. In the case of material similarity, however, when I identify a car as mine, no other will do, no matter how similar (ibid pp. 150-54).

Mathematical Perspectives on the Sign

Langue may be seen characterised in mathematics as a 'set of interpersonal rules and norms'. To learn mathematics is to master that system which allows us to be understood and to understand. The system is not articulated explicitly but realised implicitly in mathematical activity, or *parole*; algebra, for example, "is not what we write on paper but is something which goes on inside our heads. Notation is one way of representing algebra it is not algebra" (Hewitt 1985 p. 15). In algebra there are many forms that particular algebraic expressions can take; mathematical convention favours elegance, and brevity, but we can still analyse the expression, as a grammatical entity, without restricting freedom of expression. In mathematics we attempt to teach many rules directly, with seemingly little success at times. English is taught to young children without introducing the rules of grammar overtly and yet they seem able to recognise and correct speech inaccuracies, even in unrehearsed sentences. Pimm (1987) notes that in foreign language teaching the emphasis has swung from teaching rules to teaching communicative competence. It is clearly important to strike the right balance between achieving fluency in *parole*, and learning the rules of *la langue*. To know mathematics is not only to be able to retain, access and reproduce skills and techniques, but also to assimilate the structure sufficiently to be able to interpret signs and apply knowledge fluently in other circumstances.

The difficulties involved in interpreting mathematical signs are considerable and may be illustrated by imagining that we are attempting to interpret a sound chain in a foreign language. If the sentence is written then we get a good number of clues to help us on our way; if the sentence is spoken, the first hurdle to overcome is that of partitioning it into individual linguistic units. This latter task requires us to employ meaning: if the language, or accent, is unfamiliar, then it is unlikely that we will even be able to differentiate one signifier from the next. The comprehension of linguistic signs is problematic and can easily become a casualty of competing and contradictory reasoning in a language that must, at times, seem quite arbitrary to some of our pupils. In mathematics if I hear the word volume, for example, a sound image and associated concept are produced in my mind. The sound image, however, is only mathematical if associated with a mathematical idea, like 'cubical capacity'; rather than ideas such as 'book', or 'knob on a T.V. set'. Conversely, the mathematical concept of a circle, bisected by a line, does not produce a mathematical sign when associated with the signifier 'underground'.

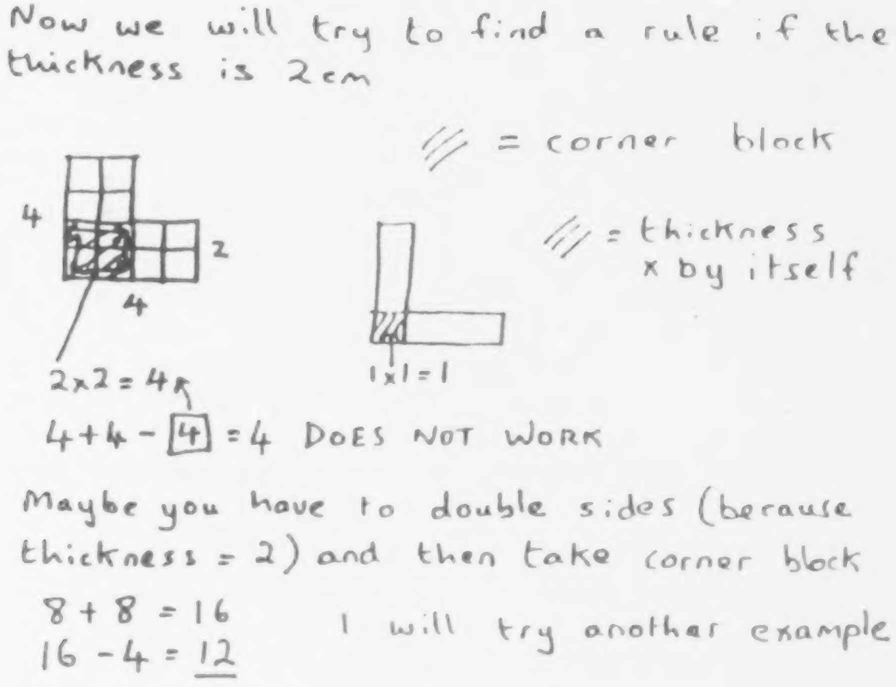
Many concepts which appear entirely natural to us, endowed with intrinsic meaning, are also arbitrary. A 'spiral', for example, at some time in our history was thought to be worthy of having its own 'label'; but there are undoubtedly societies where it does not. This does not necessarily imply that their people have different modes of perception, or thought patterns. The signifying systems of a society determine what it pays regard to and in this sense our signifieds are culturally bound. Over the centuries notation does change of course; despite collective inertia, occasionally new symbols occur and trigger the development of new ideas. In general, however, our mathematical activity is confined to the pool of symbols we inherit, and we create meaning by selecting from the symbols available to us. As Saussure remarks, "At any given period no matter how far back in time we go, a language is always an inheritance from the past" (Saussure 1983 p. 105).

Meaning is not invested in the sign itself, but is derived from its relation to other signs. Signs which contrast help to define each other reciprocally; when I describe two intersecting lines as skew, the meaning of skew is derived from its contrast to the term perpendicular. Words used to express similar notions limit each other reciprocally; hence, when I ask for the perimeter of a shape, I immediately imply that the shape is not a circle, or, I should certainly have used the term circumference. The mathematical writing system is again arbitrary and differential. The numeral 7, for example, may be written in a variety of ways, the only important thing is that it should distinguish itself from symbols like 1 with which it could easily become confused.

Relationships between signs are of supreme importance in mathematics; Pimm (1987) quotes many examples in mathematics where order, relative size, orientation and repetition are all of major significance. The signifier 6, for example, can sign in infinitely many ways: 63, 631, 6%, $6, 6!, 6" and 6'. Pimm observes that confusion arises, particularly, when entities of the same form are capable of signifying differently in the same context. He gives the example of the symbol dy/dx, the differential coefficient, which could conceivably be taken for an algebraic fraction and cancelled to y/x. The situation here is exacerbated because there are times when it is useful to manipulate the differential coefficient as if it were a fraction: the product rule dv/ds * ds/dt = dv/dt would be an example of this.

Discussing Mathematical Activity

Saussure's insights into the nature of language may illuminate our understanding of the factors involved in facilitating communication and understanding in the classroom and may provide a framework in which to analyse the processes of mathematical activity. To explore this possibility I take as my example an episode which occurred when a group of 12 year old girls were investigating the area of 'L' shapes (McNamara 1995). The students began considering 'L' shapes that were as high as they were wide and 1 cm 'thick'. They very quickly devised a formula which appeared to work and set up a powerful linguistic and symbolic representation of the situation. The formula, often referred to by the girls as, "Side plus side take corner block", was further validated when it was found to work on 'L' shapes with sides of different lengths, but still 1 cm in thickness. One member of the group, Joanne, observed that: "The formula also works when the sides are different to each other. No matter what the length of the sides are they still have a sharing block as long as the thickness is 1 centimetre." When attempting to apply the rule to an 'L' shape with thickness 2 cm, however, difficulties soon emerged as is seen in Fiure 1 below where the account is again related by Joanne.


4
2
4
2x2 = 4
1x1 = 1

Now we will try to find a rule if the thickness is 2cm

▨ = corner block

▨ = thickness x by itself

$4+4-\boxed{4}=4$ DOES NOT WORK

Maybe you have to double sides (because thickness = 2) and then take corner block

$8+8=16$
$16-4=\underline{12}$

I will try another example

FIG 1

The problems here are generated, I conjecture, largely as a result of the group's verbal construction of the problem: it is, I feel, limited and persisting beyond its useful life. The formula "Side plus side minus corner block" was not, in my opinion, an inert form of words representing the referent. Saussure perceived that speakers' linguistic concepts were not simply "private pictorial images of corresponding things in the external world" (Harris 1987 p. 61). The expression, "Side plus side", appeared to have focused the girls attention on the length, rather than the area of the sides; as a consequence it appeared that the girls were regarding the 'L' shape as one dimensional. Except, that is, in one respect, the signifier: "corner block" denoted very clearly the two dimensional image of a "thickness x thickness" square. Saussure would most probably refer to this as a relatively motivated signifier which is associated naturally with its particular concept. The word parallelogram would be another classic example of the exceptions which to him prove the rule of arbitrariness. Harris is not convinced, he believes that Saussure has actually misrecognised systematicity in linguistic structure for reduced arbitrariness; he offers as an example: "... the fact that the price of a loaf of bread is arbitrarily fixed at five francs does not mean that charging two and a half francs for half a loaf is only relatively arbitrary. (ibid p. 133)

When the formula mentioned above was applied to obtain the area of the new 'L' shape with thickness 2 cm, although the signifier "corner block" allowed the group to accommodate to the change in variables, "Side plus side" did not have the necessary flexibility. On this occasion the error was quickly detected; although no record survives it is probable that the group were carefully checking their results by counting the squares which were outlined on the diagrams. This strategy quickly lead to Joanne's realisation that the formula "Does not work", and the conjecture "Maybe you have to double the sides (because the thickness =2)".

This latter remark is interesting in that it still does not overtly indicate that Joanne has recognised that the 'sides' of the 'L' shape are two-dimensional. The explanation she proffered to support her conjecture that the sides must be doubled, "Because the thickness = 2", seems more to do with the vagaries of numbers than a traditional length times width area sum. In fact, nowhere in their account of the investigation do the group use signifiers, or methods conventionally associated with area. There is even, remarkably for Joanne, a total lack of units of area on the two pages of work. The sides of all the diagrams are meticulously labelled in 'cm' and the first two areas are allocated the same linear dimension but after that they appear abandoned in a dimensionless limbo.

Later in the investigation Joanne and Kay decided to put their spoken formula into algebraic expressions and in the attempt discovered that Joanne's formula "s plus s in brackets times t minus c" (where c represented the area of the corner block) did not have the versatility of Kay's formula "l plus h times t minus c." Joanne's formula did not, they discovered, allow for the sides of the shape to be different lengths. Joanne thus learnt of the limitations of a mathematical variable, gaining access through this activity to 'mathematical *langue*'.

Saussure defined *langue* and writing as two separate systems of signs. Writing, it is generally accepted, is semiologically secondary and existing only to represent the speech (Lyons 1972, cites phylogenetic, ontogenic, functional, structural and learning priorities of speech over writing). Writing has, in the opinion of Saussure, achieved unwarranted prestige owing to the clarity and permanence of its image, the inconsistencies between pronunciation and spelling, and the importance of literature in a developed society. Saussure, however, does make one interesting exception to his rule and that is in the case of ideographic writing, where he concedes that a symbol may act as a signifier and "... represent the entire word as a whole, and hence indirectly the idea expressed" (Saussure 1983 p. 47). Mathematics is thought by many to reverse the natural order in favour of the primacy of writing; it seems that it may, by virtue of the privilege allowed to ideographic languages, assert a position as a mixed semiological system. Undue emphasis upon the written, however, may condemn mathematics, along with other 'dead' languages, to the pages of a book and the mathematician to a rather solitary existence. The underlying philosophical belief is still apparent in many mathematics classrooms today, where mathematics is learnt in silent commune between the student and his text book or individualised work scheme. This strategy, if consistently employed over a period of time, would potentially deny the student easy access to a major source of reflective knowledge generated by interaction with peers; knowledge such as that acquired by Joanne in the episode related in the last paragraph.

In their algebraic formulae the students had developed, on their own initiative, a perfectly adequate notational representation, albeit in inherited symbols and punctuation. Although tremendously impressed by their efforts, however, I cannot suppress an underlying dissatisfaction and am compelled to attempt to modify their notation. I immediately recognise that the variable 'c' can be eliminated from the equations and replaced by 't^2'. My mathematical training has taught me to favour elegance, brevity and simplicity in an algebraic formula; I cannot resist the opportunity of initiating the girls into conventional ways of couching algebraic statements. My reaction is automatic and I do not evaluate in my mind the relative merits of the available options. I cite, as my reason for wishing to eliminate 'c', that it introduces another letter which might be confusing. There are at present four letters in the formula, all chosen carefully to embody a particular meaning. I wonder if anyone, apart from me, is convinced of the pressing need to reduce that to three? Does this appear just an arbitrary whim of mine? The letter which I am proposing to eliminate has, in fact, been the lynch pin of the entire investigation. Why did I pick on the 'c"? Kaput (1991 p. 55) reflects upon how our notation system organises our mathematical experience he advocates more designer input into functional mathematics notation. Unwittingly I have endangered the group's ownership of the algebra which they developed, and the particular meaning it held for them. It is perhaps an ever present tension between inculcating the student into the conventions of algebra, and recognising their right to freedom of expression, where it is analysable as grammatically correct, using the algebraic rule system.

Of all our semiological systems, language is of prime importance in the construction of reality. Words are not labels attached to pregiven concepts, languages which are the "... collective products of social interaction, supply the essential conceptual frameworks for men's analysis of reality and, simultaneously, the verbal equipment for their description of it. The concepts we use are creations of the language we speak" (Harris 1987 p. ix). Language is the medium through which, and in which, the student's mathematical ideas are formed and exchanged. The study of language, however, is more problematic in important ways to the study of other sciences because, whereas in science the object of study is distinct from the language of description, in linguistics it is the perspective adopted which creates the object (Saussure 1983 p. 23).

It is clear then, that we cannot delimit the language of learning from what is learnt or from the viewpoint embraced. It is too restrictive a remit to ask whether mathematics is, or is not, a language. Mathematical constructs and the language in which they are conceived are inextricably linked within mathematical activity. We must move away from seeing the application of language to a situation simply as a labelling process, and must explore further the linguistics mechanisms through which the mathematics our students encounter comes into existence.

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STUDENTS' EARLY ALGEBRAIC ACTIVITY: SENSE MAKING AND THE PRODUCTION OF MEANINGS IN MATHEMATICS

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ABSTRACT

The main goal of this paper is to present a psychological perspective of the development of meaning in algebra, from a point of view where the relations between signifiers and the signified in mathematics are seen as reciprocally constituted in a learner's activity. The concept of algebraic activity is suggested and illustrated through a brief case study that focuses on early algebra students' competence at building idiosyncratic but powerful meanings for algebra.

INTRODUCTION

The main goal of this paper is to present a psychological perspective of the development of meaning in algebra, from a point of view where the relations between the concrete and the abstract, and between signifiers and the signified in mathematics are seen as reciprocally constituted in a learner's activity, as he or she continuously produces and negotiates meanings through interaction and communication within specific cultural practices. Such a view diverge substantially from the classical discourse and methods of algebra research and instruction, where processes of mathematization (supposedly required to move from the concrete to the abstract) are seen as governed by rules whose correct application presumes (or implies) *suspension of sense making* (see Schoenfeld, 1991). As it is, the problem of meaning is rarely discussed or referred to in traditional texts, where processes of symbolic manipulation and mathematical rigor are overemphasized.

The problem of meaning, however, is a fundamental one for Psychology and for the Psychology of Mathematics Education. Thom (1973, cited in Otte & Seeger, 1994), for example, affirms that "the real problem which confronts mathematics teaching is not that of rigor, but the problem of the development of 'meaning', of the 'existence' of mathematical objects." (p. 202) The perspective presented in this paper assumes that, if we take any conceptual entity as developed within specific social practices (which themselves create specific types of relations among their participants; Walkerdine, 1988), to produce meaning is equivalent to create relations among conceptual fields (Vergnaud, 1990), mediational tools (Werstch, 1991; Vygotsky, 1978; Meira, 1995a), and activities (Leontiev, 1981). This view, which I will briefly discuss below, informs a psychological perspective of meaning used in this paper to investigate students' developing understanding of algebra, in particular as a specialized language (with its own notational system and semantics) to explore particular worlds and to talk about them.

Algebra, algebraic activity, and the problem of meaning. An extensive body of research in algebra instruction and learning, not to be reviewed here, has demonstrated teachers' and students' difficulties regarding this subject (e.g., NCTM, 1988). These difficulties are partly due to traditional instructional approaches that focus on various aspects of algebra as if they were devoid of meaning (as it is sometimes supposed about the process of manipulating symbols on paper), restraining children's understanding of the conceptual and representational objects that make this domain meaningful. Although there have been many studies about algebraic thinking and the difficulties involved in developing it in children, we lack a robust psychological understanding of students' *algebraic activity* and the ways they generate meaning for algebra problems. The concept of algebraic activity is explored here in connection with Leontiev's (1981) Activity Theory. This approach considers thinking itself as intricately related to one's motivated actions, in such a way that the social and material organization of specific practices and situations are an essential aspect of mathematical sense making. In such framework, meaning is seen not as a cognitive product, but as a socio-historical accomplishment of communities of practice (Lave & Wenger, 1991). Meaning is thus (1) inherently social; (2) an emerging phenomenon situated in activities; and (3) intimately dependent on interactional and material resources of particular situations. A basic assumption here is that all actions have a semantic content, that is, they have significance and influence by virtue of the meanings they acquire in specific sociocultural contexts (Meira, 1995b, p. 277).

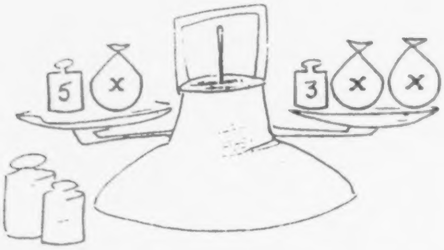
Building on this perspective, I propose the concept of algebraic activity as a descriptor of actions that involve, necessarily but not exclusively, a clear intention (or motive) of using knowledge of algebra as a means to accomplish and justify responses to mathematical problems and/or to communicate mathematical results and processes. For example, the production of algebraic notations during problem solving engages the individual in algebraic activity in the sense that he or she begins to share with other members of certain communities (e.g., the classroom or the world of professional mathematics) a particular discourse about mathematical problems. Using notations is obviously only part of the very complex process of thinking algebraically (sometimes an unnecessary part), but that indicates and supports the individuals' insertion in certain discursive practices that are critical for one's participation and access to mathematics and, in particular, to algebra. In this respect, using algebraic notations as part of a language connects the individual to "the spoken language of the mathematics classrooms"; to "the use of particular words for mathematical ends"; to "the language of [mathematical] texts"; and also to "the language of written symbolic forms." (Pimm, 1994, p. 159)

It is critical to note that this view does not limit algebra to the use of its notational system, nor algebraic activity to the meanings intended by experts. The situation is similar to the young child that mumbles words in very simple sentences without completeness or syntactical correction, but plenty of meaning for the communication being attempted with an adult or a peer. If a child says "wa-wa", is that English? From a strictly formal point of view, the answer is no. However, if we take the context of its use, such as the presence of a pointing gesture to a sink faucet and/or of someone acquainted with that child's vocabulary, we may attribute to his or her attempt the intention of communicating a need for tap water ("Please give me water" ≈ "wa-wa"). In the same sense, the student who is unable to fully and competently use algebra as intended by experts may be engaged in algebraic activity as he or she uses algebraic notation, follows certain routines of action and shares certain premises of communication which are part of the algebraic discourse in the classroom (such as moving unknowns between the terms of an equation). He or she is producing meaning for algebra, and is thus involved in algebraic activity.

The case study that follows attempts to illustrate the concept of algebraic activity, and the underlying premise that, from a psychological perspective, all action involves the production of meaning. More specifically, I take on the questions of what meanings students create during algebraic activity, and how are these meanings related to the development of sense-making in this mathematical domain.

A CASE STUDY ON THE PRODUCTION OF MEANINGS IN ALGEBRA

This illustrative study explores the problem solving activity of one pair of seventh grade students, and their developing understanding of algebraic equivalencies and the mechanisms of "symbolic manipulation". The students (S and T), aged 13, were volunteers from the same classroom in a public school in Recife, Brazil. These students had little initial knowledge of formal algebra. Although they had some practice with solving equations in one variable and systems of equations, observations of their classroom showed that instruction on algebra was limited to the presentation of procedures to solve problems about "situations" previously modeled with algebra by the teacher or the textbook. During the interviews for this study, the students were given a paper drawing of a two-plate scale, several individual drawings of weights and bags marked "x", and problems about balancing the scale. Similar setups have been studied by Vergnaud (1986) and Carraher & Schliemann (1987), who argue that the two-plate scale may be adequately used as a physical referent for children to develop the concepts of equivalence and unknown. (My use of the scale metaphor aimed at eliciting students' algebraic activity, rather than arguing for its use as an instructional device.) The figure below reproduces the drawings presented to the students (the drawings of bags and weights were made in individual pieces of paper to allow possible handling of these representations).



Sample problem: How much should the bags in the drawing weigh to keep the scale balanced?

Many of the problems about the scale were correctly solved by S and T (and other pairs of students not reported here) through a "test of hypotheses" (Carraher & Schliemann, 1987) where numerical values are attributed to the unknowns ("x") and the equality between the plates is checked arithmetically on the representation. But the students also attempted to model the arrangements of weights and bags on the scale with algebra, even when an arithmetic solution had already produced a correct answer. The fact that these students were generally unable to build adequate algebraic models of the arrangements is not surprising, given the nature of their classroom experience. Nevertheless, the equations written by the students were not at all devoid of meaning. In fact, some of these equations were consistently manipulated so as to produce the same result originated by what the students called "logical answers" (the result of testing hypotheses).

The discussion challenges the so-called "suspension of sense making" in algebraic activity, and attempt to show that meaning is continuously pursued by early algebra students during problem solving, despite the school emphasis on syntactical correctness. The analysis focuses on the students' competence at building idiosyncratic but powerful meanings for algebra, rather than on their inability to think algebraically or to deal with algebraic structures. This competence is rarely explored in school, either in respect to building arithmetical interpretations for algebra problems, or in respect to taking algebra as a tool to think with. From this perspective, I analyze the ways in which the language of algebra (in particular its notational system) were transformed to fit what the students intuitively knew about the scale, and the types of answers they expected.

The first problem, which presented two bags of the same but unknown weight in the left-hand plate and a weight of 4kg in the right-hand plate (or "x+x=4"), was a training problem easily solved by the students without explicit use of algebraic notation. The following passage shows their solution, elaborated right after the arrangement of weights and bags on the drawing of the scale is shown:

Episode 1

Time	Scale: x x — 4
0:00:00	Int: *I want to know how much each bag weighs*
	S: *Two kilos.*
	T: *[Two kilos.*
	Int: *How did you do it?*
0:00:09	T: *(Pointing to the plates on the representation) Two plus two, four.*

In the passage above we see that the presence of unknowns in the representation of the problem is not enough to trigger the use of algebraic notations or procedures. In a sense, the equation "x+x=4" can be regarded as arithmetic rather than algebraic (in accordance with the categories suggested by Filloy & Rojano, 1989). The students' approach to the second problem, for which an expert's model could be "x+x+3=x+5", involved instead a complex interplay between using algebraic notation and empirically testing hypotheses. Technically, the modeling process and the algebraic procedures employed by the students are faulty. Note in the excerpt below, however, how the results of such an inadequate use of algebra is profited by the students to make their work meaningful .

Episode 2

Time	Scale: x x 3 — x 5	Paper work
0:00:49	Int: *How much does each bag weigh?*	
	S: *To balance... (Both students observe the drawings of the scale, bags and weights very attentively.) (10 sec.)*	
	Int: *I want that you do it talking out loud, thinking out loud....*	
0:01:07	S: *Here (pointing to the right-hand plate with 5kg and one bag marked "x"), right, will be the following (brings a sheet of scratch paper towards him)... five x,*	
	on the other side (pointing to the left-hand plate with 3kg and two bags marked "x" on each) will be times three x to the second (completes the expression on paper).	5x
	Then here (right-hand plate) we do the following (turns to scratch paper)... let's pretend that here... let's pretend that here [we] isolated this one here, this x (next to 5)... Then put, five, times... no, don't put times,	$5x.3x^2$
		~~5~~.
	put five on one side equals three on the other and here you put the	5=3
	two ex's, x here and x two here (writes "$x=x^2$" right below "5=3", as in a system of equations). Now we do the following, now it will invert, bring the three here (with the term "5=), oh, the five there	$x=x^2$

	(with the term "$=3$"), and the x that was here ("x^2") bring here	
	(with the term "$x=$")... Then add, put everything together, makes x	
	times x two equal three minus... it's minus here ("$x-x^2$" instead of	~~$x.x^2=3-$~~
	"$x\ x^2$")... minus five... x three equals two kilos: then this means that	$x-x^2=3-5$
	each bag (pointing to bags on the scale) has two kilos (pointing to	$x^3=2$
	the equations on paper), two kilos, then now we're going to add these	
	two (pointing to the representations of weights on the scale)... Here	
	(in the equation) we already now that x is equal two; three... plus five	
	is equal eight. Eight... (Looks attentively and alternatively at the	~~$3+5=8$~~
0:03:21	*scratch paper and the configuration of the scale. Scratches over this*	
	last step.) This is not going to work...	
	T- *[No*	
	S- *He is just asking for the result of the bags (while pointing to the scale), right?*	
	Int: *Yeah, how much does each bag weigh?*	
	S: *Two kilos.*	
0:03:29	Int: *How did you do to find it?*	
	S: *I put, moved the x, did that thing of first member and second member (waving his hand to indicate transference of elements between the terms of the equation), that process, after that I added the x, it gave three x, three to the, uh, x to three, equals to, subtracted three from five, got two, then x equals two, each of these ex's (still pointing to the equations on paper) is equal to two (now pointing to the plates in*	
0:03:59	*the scale), has two kilos.*	

In the passage above the students' attempt to use algebra resembles a language game of "pretending", as S himself noticed ("let's pretend that here [we] isolated this one..."). Given the first expression written on paper ("$5x.3x^2$") and the procedures employed in this game, it is at first surprising that the students achieved an answer to the problem (a "correct" one!). In fact, we can hardly say that algebra was used at all. However, it is reasonable to suppose that S could intuitively recognize the appropriate answer to the problem (e.g., he prefers the incorrect "$3-5=2$" which produces a "good answer", over the arithmetically correct "$3+5=8$" which "*is not going to work*"), and that he played a pretending game with the symbols on paper in order to turn his intuitive expectations into an "algebraic result". It is also reasonable to suppose that the answer "$x=2$" (that is, "$x^3=2$") was reached entirely by accident. Nonetheless, the students attribute to this answer the status of a correct and meaningful response to the problem at hand. In the dialogue below, while presenting a simpler "logical" way to resolve the problem, the students build an empirical justification for that result.

Episode 2 (cont'd)

Time Scale: x x 3 — x 5

0:04:00 T: *But [you] could also do it in a simpler way.*

S: *Yeah, could make it simpler...*

T: *Because this one here (pointing to the bag in the right-hand plate)/*

S: *[Add seven (pointing to the right-hand plate), seven (pointing to the left-hand plate, and looking at the partner)...*

T: *Yeah, if here (bag in right-hand plate) were three (summing up 8), here (pointing to bags in the left-hand plate) couldn't be five, no way, because the bags are equal (e.g., there could not be a bag of 3 plus another bag of 2)... if it were three and two here* *(pointing to each of the two bags in the left-hand plate), it couldn't be.*

Int: *And how do you do to find what number to use?... If one doesn't do it this way (pointing to the expressions on paper), is there a way?*

0:04:22 S: *I think you would add the weights, right? The weights, then you'd pass, could do like this (handles the drawings of bags and weights, placing the weights of 5 and 3 together in the right-hand plate, and all the bags together in the left-hand plate), would sum up these one (the weights), eight, to balance it must be, here was eight (right-hand plate), two, four, six (adding the three bags now in the left-hand plate), it doesn't work.*

T: *[It doesn't work this way.*

S: *You're right.*

T: *Each bag weighs two kilos.*

S: *Then [you] would do as I did in the head, from the start, using the logic (returns weights and bags to original positions). If each bag weighs the same, for this (right-hand plate) to be balanced with this (left-hand plate), if this was three (bag on the right-hand plate), three kilos would be eight (5+3=8). For this to be eight (total value on the left-hand plate), one bag would have to weigh more than the other, because putting here two and three, it gives five, here would have to weigh three (3+2+3=8) in any case, one (bag on the left-hand plate) would have to weigh more than the other one kilo more.*

Int: *And doing this other way (pointing to equations), do you get it right?*

0:05:39 S: *Yeah, it gives two kilos for each bag.*

The test of hypotheses is used above as a method of justifying empirically the result given by the equations ("*if here were three, here couldn't be five*" and "*if it were three and two here, it couldn't be*"). In the sequence, there is evidence that at least part of the students' mistakes in algebra may be consequence of misunderstandings regarding how the scale works. For example, the students equate the sum of bags to the sum of weights (disregarding their position on the scale), much in the same way they put like terms together when writing the equations. However, another problem remains which is related to the format of the answer on paper: $x^3=2$ rather than $x=2$. The conversation below, initiated by the interviewer, reveals a different sense for "meaningless symbol manipulation".

Episode 2 (cont'd)

Time	Scale: x x 3 ↔ x 5	Paper work
0:05:54	Int: *I'm in trouble here (pointing to the scratch paper), because it shows x to the third equals two (pointing to "$x^3=2$"). Then, it's not the x that equals two, it's x to the third equals two...*	
	S: *No, it's the x. The three, we move it here (as a "power rate" to the second member), there it wouldn't be a power rate anymore, but a sum, putting two kilos for each bag...*	~~$x^3=2$~~ $x=2^3$
	Int: *What does the (power of) three show?*	
0:06:35	S: *The bag, the number for x, the number of bags wanted!*	

Again, from a strictly formal point of view which takes algebra as a closed mathematical domain the passages above may appear meaningless. If instead one takes activity (not algebra or algebraic thinking) as the focus of analysis, a much different picture emerges. From this perspective, these students were (as people always are) engaged in a goal-directed activity in the context of which their actions become meaningful. For instance, S is all too ready to transform the expression "$x^3=2$" into "$x=2^3$" (no longer a power index but the number of bags in the physical arrangement) as long as this action sustains a meaningful connection between his use of algebra and the world it is supposed to model.

CONCLUDING REMARKS

The perspective presented in this paper questions the traditional view of problem solving in algebra as a stepwise process where symbolic manipulation is often seen as devoid of meaning. The ideal (perhaps desired) situation where an algebraic model is manipulated without reference to the modeled world, and more generally to the social and material contexts of problem solving, may be an impalpable psychological purism and is certainly not applicable to apprentices of an unfamiliar knowledge domain. There seems to be no place for such things as "suspension of sense making" when early algebra students handle symbols and languages in activity. It is certainly plausible to think of professional mathematicians as handling complex algebraic expressions in a purely syntactical manner. But even then, there may be certain aesthetic values that will keep the expert from turning himself or herself into a reckoning device. In this case, the aesthetic values attributed to the procedures followed to handle the syntax give the essential semantics to the actions performed.

This paper showed one example where the students were generally unable to "think algebraically" (since there is little evidence that they could understand equations in terms of general structures), but were very much involved in "algebraic activity" --in the sense that algebraic notations and procedures were used to pursue specific goals and to get things done. Thus, taking activity (with all that entails) rather than thinking as the unity of analysis, we come to a psychological/epistemological perspective where the production of meaning is ubiquitous to doing mathematics, not a possible byproduct. A central aspect of this perspective is the stress in the so called "negotiation of meaning", bringing forth a tension between mathematical conventions and formalisms, and the lively deconstruction and reconstruction of mathematical meanings during problem solving activity.

The observations made here are in close connection with a view of algebra instruction where students are encouraged to reflect on their (joint) activity and to make those reflections publicly available. For that to happen, algebra ought to be seen as a tool for leaning rather than as a source of direct, unmediated knowledge. Given the exploratory power of its language and the multitude of concepts embodied in its structure, the idea of algebra-as-a-tool suggests a renewed set of assumptions regarding early algebra instruction. Firstly, there is no unique class of approaches that will safely conduct students away from major difficulties in understanding this subject matter. According to Lins (1994), for example, the situations described by two-plate scales may be good enough to deal with equations such as "$3x+10=100$" but the scale metaphor is not appropriate as a physical instantiation of equations such as "$3x+100=10$", since it cannot properly embody operations with negative values. Second, modeling tasks are not an instructional panacea. The idea that using physical devices and situations can provide "meaning" for mathematical concepts is misleading for it abstracts away a careful analysis of the activities and contexts in which such artifacts are made to signify. I have previously questioned the inadequacy of traditionally assumed dichotomies which define mathematical sense making as *abstract* reasoning and opposes it to *practical/concrete* actions: "Making sense of instructional devices go beyond having certain cognitive competencies, to involve 'extra-cognitive' processes such as the mutual appropriation of goals through social interaction as well as the dialectical interplay between knowledge and material tools." (Meira, in press) Third, tasks will always be transformed by teachers and students in activity, in such a way that they will produce their own (individual and collective) meanings. This does not mean that we should not care about what tasks to prescribe, but that they have no fixed meanings. Moreover, it means that students can be led to make useful approximations between the idiosyncratic meanings they create with those intended by the expert. The concept of "readiness" (to understand the expert's meaning) could thus be replaced by that of "susceptibility" for learning, pretty much in the sense implied by Vygotsky's (1978) concept of Zone of Proximal Development. In a related manner, this research calls attention to the richness of students' errors in algebra and the possibilities of preparing teachers to better understand the web of meanings underlying such errors.

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PERFORMANCE AND UNDERSTANDING: A CLOSER LOOK AT COMPARISON WORD PROBLEMS

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This work deals with second graders performance and understanding in solving compare problems. Children are asked to solve all six types of compare problems and explain their solutions. A large proportion of children who solve a given problem correctly give incorrect explanations. On the other hand, a large proportion of children who give an incorrect solution exhibit a partial understanding of comparison situations. Additional information about their knowledge is obtained from their solutions in context free situations.

INTRODUCTION

Following a large body of research on additive word problems, Riley, Greeno, and Heller (1983) in their work extended by Nesher, Greeno, and Riley (1982) suggest a psychological developmental theory for knowledge related to word problems. The theory was built to explain why some problems can be solved by very young children, while others can be solved only at a later age.

According to their analysis children can solve additive comparison problems (termed: 'compare problems'), which ask about the difference or about the compared group at the part-part-whole stage. Compare problems which ask about the reference set require a higher stage, -specifically, the child has to perceive the order relation as a two directional inequality.

The purpose of our research is to make a more detailed description of the child's knowledge. Children's performance in all six types of compare problems is observed and the children are asked to explain their answers. As a result, it is possible to say more about those who fail and to investigate whether a child who gives a correct answer really understands the problem's structure.

PROCEDURE

Second grade children from 2 classes (n=38) were asked to solve all six types of additive comparison problems. First, each child solved six compare problems, one of each type. Later, the children were individually interviewed. During the interview they were asked to solve problems that involved non-contextual situations describing relations between sets, which correspond to the six compare problems. Then they solved each of the written (contextual) problems again and were asked to explain the solution process.

RESULTS

For each problem the following values were calculated:
The proportion of correct answers, the proportion of correct explanation for the correct and incorrect answers, the proportion of correct answers for context free situations within each of the subgroups (performance , explanation). In the presentation we will detail the three dimensional data table. In this paper only a part of the existing data is presented.

Table 1 details the proportion of correct answers for each of the six compare problems (C1 - C6) together with the proportion of correct explanations in this subgroup.

Table 1

Percentages of correct answers and correct explanations for all six types of compare problems.

Problem type	Correct performance	Correct explanation
C1	63	39
C2	73	52
C3	63	27
C4	62	22
C5	39	18
C6	63	21

Note: The percentage of correct explanations for a given problem is calculated for the subgroup of correct answers to this problem. However, the percentages in each column are of the total number of children who answered a given problem.

For each of the six problems presented again during the interview, an analysis of the different explanations has been made. The explanations have been categorized according to their content and an effort has been made to identify the developmental level of each answer on the range suggested by Nesher et al (1982). The developmental level has not always been relevant and therefore also not always determined. This happened when children did not exhibit any effort to construct an image of the situation. For example, some of them turned immediately to a verbal cue and used it to decide which direct operation to use.

The following answers are examples of explanations given for problem C5 (compare 5).

The problem: *Dan has 5 books.*
Dan has 3 books more than John.
How many books does John have?

Answer 1 (a correct answer): 5-3=2

Correct explanation: *Dan has more books and John has 3 books less than Dan, so John has 2 books.*

Incorrect explanations:

1. *Dan had 5 books and now he has 3 books. This means that he gave John 2 books. So John has 2 books.*
2. *Dan has 5 books. Dan has 3 books. You subtract to find by how much 5 is more than 3.*
3. *You subtract because 3 is less than 5.*

Answer 2 (an incorrect answer): 5+3=8

Incorrect explanations:

1. *Dan has 5. Dan has 3. Together he has 8.*
2. *John has 3 more than Dan, so John has 8.*
3. *I added because it says 'more'.*
4. *I added because you always add.*

Answer 3 (an incorrect answer): John has 10.

Incorrect explanation:

1. *Dan has 5. Dan has 3. Together he has 8.*
   John has more than Dan. He might have 10.

Answer 4 (an incorrect answer): You can't solve it.

Incorrect explanations:

1a. *Dan has 5. Dan has 3. Maybe it's another Dan.*
You can't tell how much John has.
1b. *Dan has 5. Dan has 3. They want to confuse me.*
You can't tell how much John has.

Although many of the explanations are incorrect and involve the transformation of a compare 5 problem into a simpler problem, still the type of invented problem and its solution indicates, in some cases that the child has some understanding of a comparison situation. For example, a child who gave the second incorrect explanation for answer 1 shows that she knows how to compare two given amounts. A child who gave the second incorrect explanation for answer 2 shows that she can solve compare 3 problems, where one has to calculate the compared set. These two children perform at level 3 (part-part-whole).

Additional information about the child's knowledge is deduced from the performance in context free problems. The context free problems deal with the relations between sets in a way that corresponds to the six compare problems. For example, the child is asked to build a set of objects which has a certain (given) number of objects more than the number of objects of another (given) set. This request is a context free situation which corresponds to a compare 3 problem. These situations involve knowledge which can be considered a prerequisite for performing the corresponding compare problems. Table 2 shows the percentage of children who performed correctly in the context free situations although they did not give a correct explanation.

Table 2

Percentages of incorrect explanations and correct context free performance for each of the six compare problems.

Problem type	Incorrect explanation	Correct context free
C1	60	33
C2	48	21
C3	72	26
C4	78	44
C5	81	18
C6	79	9

Note: The correct context free responses in this table are identified within the set of children who gave an incorrect explanation. The percentages are calculated as a proportion of the total number of children who answered a given problem.

The details of these context free situations together with the specification of examples of children's performance will be further elaborated in the presentation.

DISCUSSION

The findings lead to several observations with regard to the comparison of the child's performance with her explanation (taken to indicate amount of understanding), and with regard to the comparison of the child's ability to handle context free situation with her ability to understand a given problem. The main points are:

1. A large proportion of students who supposedly give a correct answer, have arrived at this answer by using an incorrect analysis of the situation.
2. Children who give an incorrect answer might have a partial understanding of the comparison situation.
3. Some children can analyze the set structure in a given problem type correctly as long as the problem involves context free set relations.

Asking the child to elaborate on the way she solves a given problem enables us to observe two steps in the process: a. The way the child perceives the problem. b. The way the perceived problem is solved.

Verschaffel (1994) investigates the problems' encoding stage by asking children to retell the problems. He deals with four of the six compare problems in which the unknown is one of the two compared sets, as these problems are relevant for checking the consistency model. According to the consistency model the child expects, after being told about the quantity of one set, to hear how the other set relates to it.
The children in Verschaffel's study are fifth graders. Still, many of them convert an inconsistent compare problem (compare 5 or compare 6) into a consistent problem (usually compare 3 or compare 4), sometimes making a correct and sometimes an incorrect conversion.

The children in our study are much younger (second graders), therefore -it is not surprising that they convert a given compare problem into a non compare problem. Sometimes the conversion is made into a change problem, and sometimes into a simpler (even trivial) problem. The interview enables us to detail the different kinds of problems into which a given problem is converted. It also enables us to see how the problem is then handled. These observations give us more information about the child's understanding of the different situations. For example, a child might incorrectly convert a compare 5 problem into a compare 3 problem (keeping the word "more" instead of switching to "less" to get a correct conversion into a consistent problem). However, this child might then solve the new compare 3 problem correctly, showing that she has a partial understanding of comparison situations, and also indicating that she is able of performing a task which, according to Nesher et al (1982) requires that the child is at level 3 (holding a part/part/whole schema).

It is interesting to note that a child that has answered a given problem correctly might, in fact, know less about comparison situations than a child who answers the problem incorrectly. These findings support the claim that children's performance should not be judged in correct/incorrect terms, and show that even incorrect performance can tell us a lot about what the child does know.

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GRAPHING CALCULATORS AND PRE-CALCULUS: AN EXPLORATION OF SOME ASPECTS OF STUDENTS' UNDERSTANDING

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Abstract

This paper presents a summary of the work done in an exploration with students of a precalculus course concerning the influence exerted by the graphics calculator on their understanding of the function concept, attending to the operational-structural duality of the conceptions related to it (Sfard, 1991) and to the use of notation systems (Kaput, 1992). A quasi-experimental study showed no evidence of influence, but the whole process gave new information about the implications of the resource in our classrooms.

Introduction

The research project "Students' Learning and Understanding and the Graphing Calculator" is part of the research program, "Graphing Calculators and Pre-calculus" developed by the research group of "una empresa docente" during 1993 and 1995.[1] This paper will give an overview of the project, which worked on the exploration of students' learning and understanding of some topics related to the function concept. The theoretical framework was build upon three aspects: the internal representations of knowledge, proposed by Hiebert and Carpenter (1992); the dual nature (operational–structural) of mathematical conceptions,[2] proposed by Sfard (1991); and the relationship between notation systems and school mathematical activity, proposed by Kaput (1992).

Framework

It is widely accepted that knowledge is internally organized in networks of nodes and links, the nodes seen as facts or procedures, and the links seen as 'relationships' between the nodes (Hiebert & Carpenter, 1992, pp. 66). Understanding can be considered as the addition of a new node to the existing individual's knowledge network or as a reorganization of that network by using relations of similarity or difference or by using inclusion relationships (Hiebert and Carpenter, 1992, pp. 67-69). Sfard points out that some mathematical concepts may be seen as objects and as procedures,[3] and that for a deep understanding of the mathematical concepts, the individual has to develop the ability to use the two views of the concept, the structural view (i. e., as an object), and the operational view (i. e., as a procedure).

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[1] The main goal of the program was the exploration of the influence of the graphics calculator on curriculum design, on the class interaction among the teacher and the students and with mathematics, on the teacher's beliefs, on students' learning and understanding and on the students' attitude towards mathematics. The program received support from Colciencias, Texas Instruments, the Comisión para el Avance de la Ciencia y la Tecnología del Banco de la República, and the PLACEM, Proyecto Latinoamericano de Calculadoras en Educación Matemática. The PLACEM has members from the following countries: Argentina, Brazil, Chile, Colombia, Costa Rica, Mexico, USA, and Venezuela.

[2] By conception we mean the set of internal representations and the corresponding associations that a mathematical concept evokes in the individual (Sfard, 1991, p. 3).

[3] Static structure, existing somewhere in space and time, versus a dynamic, sequential, and detailed structure (Sfard, 1991, p. 4).

Sfard shows that it is possible to speak about a continuum, from the operational view to the structural view, of the individual's learning and understanding of a mathematical concept, and that it is possible to go from one pole to the other in three steps: interiorization, condensation, and reification. Finally, she points out that "transition from processes to abstract objects enhances our sense of understanding mathematics" (Sfard, 1991, p. 29). Kaput (1992) argues that as it is impossible to see what happens in the individual's mind, the researcher has to analyze the operations he or she executes in the physical world. So, we need to suppose that to project the operations the individual is doing in his or her mind, he or she will need to execute some actions in the physical world using a specific language. Kaput defines the language as the notation system: a system of rules for identifying, creating, operating on, and determining relationships among characters; the 'characters' can be letters, numbers, graphs, or physical objects. He stresses that there exists a relationship between the use an individual gives to the notation system and the different kinds of mathematical activities done in school mathematics:

- Syntactically constrained transformations exclusively within one notation system, with or without reference to any external meanings,
- Translations between notation systems,
- Constructing and testing mathematical models, which amount to translation between aspects of situations and notation systems,
- Consolidation or crystallization of relationships and or processes into conceptual objects or 'cognitive entities' that can be used in relationships or processes at a higher level of organization. (Kaput, 1992, p. 524-525)

Kaput points out that the first type of activity strongly dominates school mathematics and that the fourth type "as a source of mathematical-meaning building, has longer-term effects, because it can lead the individual to the use of cognitive objects, instead of using them as processes (counting, taking-part, transforming)" (p 525). It is important to say that the second and third types of mathematical activities explicitly imply a 'horizontal' mathematical growth, whereas the first and the fourth lead to a 'vertical' mathematical growth.

There seems to be something common between the two last approaches proposed: the two of them give specific ideas for analyzing the similarity-difference and inclusion relationships that could take place in the individual's mind (see Figure 1):

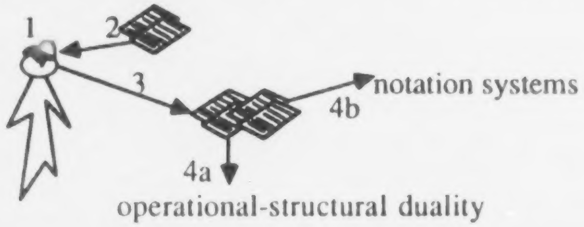

Figure 1: A model for studying an individual's understanding.

(1) In the individual's mind there is an organization of his or her knowledge. When exposed to some mathematical activity (2), he or she uses internally some language for doing the manipulations needed for working on it. In order to give a response, the individual needs to do some physical operations (3): speech, written text, or some other specific actions. But then, we can study the way in which the individual uses the notation systems (4b) and the way he or she approaches the concepts involved in the discourse or on paper, –that is, structurally or operationally (4a)– and doing so, we can have an idea of the individual's understanding of the concepts involved in the mathematical activity proposed.

The interest is, then, to see how the graphics calculator affects individual's understanding, from two perspectives: Does it help the individual develop a sense of the operational-structural duality of a concept? Does it encourage the use of several notation systems in a way that helps the individual's vertical and horizontal mathematical growth?

Methodological implications

The research on the use of graphics calculators gave us some hope about the potential of this resource in the classroom. Although that research has not shown positive results (Ruthven, 1995), there is a consensus about the importance of using an integrated curriculum, instead of using the calculator as 'just another peer': "technology-enhanced learning environments do not of themselves help students decide which features of [the mathematical entities] are the relevant ones to focus on nor how to describe their observations or conclusions [It is] the use of such technology, in conjunction with the necessary instructional support of a capable teacher, [that] can help students to objectify [those entities] to operate with and to talk about" (Kieran, 1992, p 410).

For approaching the exploration of students' understanding, we used the basic idea suggested by Figure 1: (a) giving the students some specific problems for work in an environment without the graphing calculator,(b) collecting information on the outcomes and (c) analysing the data regarding the use of the notation systems and the use of the operational-structural duality of the function concept. In order to be able to produce comparative results, we decide to (d) use the idea to a new group giving the same problems to students in a graphing calculator environment, collecting the same information, and comparing[4] the outcomes. This required two different groups of students and two different measurements of their cognitive status; one measurement was taken at the beginning and at the other at the end of a certain instructional period in which one of the groups used the calculator in their curriculum, and the other followed the standard curriculum.

To reduce differences attributed to the teacher, the same teacher taught both groups of students; and to reduce influence due to teacher's interaction with the calculator, we decided to collect data over a semester. The standard course in the second semester of 1992 was the calculator-free environment, and in the same course in the first semester of 1993, the students used graphing calculators all the time. See

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[4] We will discuss later the meaning and implications of the comparison process.

Gómez and Valero (1996) for a complete description of the procedure we followed for choosing the teacher from the 65 mathematics teachers in our department. Roughly speaking, we chose, a teacher with an instrumentalist view of mathematics, "basically a trainer" (p. 154).

For introducing calculators to the second group we developed a two pages guide for solving a system of two linear equations. That was the only instructional activity that explicitly required the use of the calculator. Although all the students were given a calculator for their personal use, there were no explicit indications in the curriculum about when or where use the resource: This was left as a personal decision for the students and for the teacher. We adopted this attitude mainly because we did not want to change the curricula: we wanted to have this 'variable' controlled.[5]

We worked on two instruments for collecting data: a test with 6 open-ended questions and an interview. We decided to give the test to each group twice a semester, at the beginning and at the end of the course and to interview four students in each group one week after each test, using a questionnaire based on the answers given to the test.[6] For producing the test, we took into account the goals, the topics covered and the notation systems that are predominant in the precalculus course.[7] This gave us a base of items for evaluating achievement and performance. We adapted questions from Ruthven (1990), Kieran (1990) and from the text of the course (Echeverry, Gómez, Gómez and Mesa, 1990). (See Mesa and Gómez, 1995, for the problems used in the test.) Even knowing that giving a standard test would make it difficult to extracting information about notation systems and about the operational-structural duality, we thought that this kind of test would reduce the students' anxiety that might arise if they were confronted to different kind of problems, and we thought that the interviews would give us more information when needed for analyzing special cases. As the university has a policy restricting the use of calculators in examinations, the students were not allowed to use a calculator in any of the tests.

Using the text of the course, we produced a set of typical –correct and incorrect– answers to each question of the test.[8] We called each of those 'the strategy' for solving the problem. For analyzing the strategy, we adapted the 'table of perspectives' (Table 1 of the Appendix) given by Moschkovich, Schoenfeld, and Arcavi (1993, p 79) and used the characterizations in each cell of the table for classifying each step of the strategy. This gave us a list of 'cells visited' for each strategy produced.

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[5] At the same time, we were exploring by ourselves the potentiality of the resource in our courses with other students. In fact, we began to accept the use of the calculators in the evaluations, a consideration that was reported as important for obtaining positive findings in the use of graphing calculators in class (Ruthven, 1990; Quesada & Maxwell, 1994). This new attitude potentially triggered a change in our views about mathematics and about mathematics teaching and learning. (Carulla, Gómez and Mesa, 1995).

[6] In this paper we are going to refer only to the outcomes of the written test.

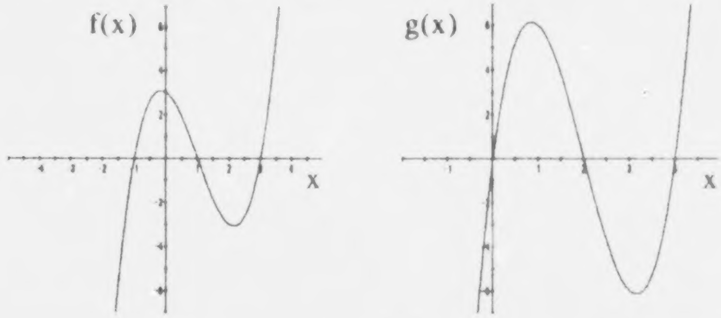
[7] The precalculus course is devoted to the study of polynomial, rational, and radical functions. Each function is treated separately covering the following topics: symbolic and graphic representation, symbolic and graphical manipulation, relationship between the tow representations, equations, inequalities, and problem solving. The last two weeks are used for working on the generalization of the function concept.

[8] Each chapter of the text has a section called "Typical Errors" and contains the errors more commonly found in the students' tests. We included answers to questions that are learned in the middle school.

Data collection and results

Even though the original plan was to analyze each question of the test, for reasons time we decided to study in depth only three problems of the test.The three exercises were chosen because at least one of the strategies used for solving them visited all cells of the table of perspectives. We will take the 6th exercise as a sample of the results found in the exploration because the results observed in this problem are similar to the results observed in the other problems even thoughit was different in style from the problems the students were used to solving. Table 2 shows the strategies posted for this exercise, the cells visited of each one, and the proportion of the students in each group[9] that used each strategy in both tests (at the beginning and at the end of the course). The text of the problem was as follows:

6. The figure shows the graph of one expression, $f(x)$ and the graph of other expression, $g(x)$, that has been obtained from $f(x)$. Write an expression for $g(x)$ in terms of $f(x)$ such that shows the transformation applied to $f(x)$ for obtaining $g(x)$. Explain your reasoning.



On the final test, the students in both groups used only the strategies that were taught in the course. They did not use the strategies learned in school. In the group that did not use the calculator, 45% of the students gave, on the final test, an expression for g, using f or a verbal description of what happened to the function; 17% gave an expression for f using a polynomial expression, and used the intersections with the x-axis for writing factors for the polynomial. No one used a trigonometric function on the final test. In the group that used the calculator, 28% of the students gave on the final test an expression for f and used this expression for writing the expression for g. This was not observed in the answers given by the group that did not have the calculators. The final answers did not show a trigonometric expression for f, but it is interesting to point out that 43% of the students used a polynomial function for describing f.

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[9] The students were freshmen in engineering, economics, business administration, and biology. We took information on 18 students in each group that took both tests.

We did not find important differences in the use of strategies attending the 'cells visited' criterion, except for the OS (Operational-Symbolic) cell. All the strategies had the OG cell[10]. In the group without calculators, 39% of the students used strategies that had the SS (Strategic-Symbolic) cell, and 33% of the students in the group with calculators used those strategies. In the group without calculators, 23% of the students used strategies that had the SG (Strategic-Graphic) cell and 33% of the students in the group with calculators used those strategies. With respect to the strategies that had the OS (Operational-Symbolic) cell, 23% of the students in the group without calculators used them, and 44% of the students with calculators used them. This large difference can be explained by the fact that the students whoused the calculator used a special approximation. They began looking for a polynomial for *f* and then did additional symbolic work writing the expression for *f* and writing the expression for *g*. According to our classification, this corresponded to operational work in the symbolic notation system. The standard curriculum gives the students opportunities to produce a rationale for the relationship between the zeroes, the intersections, and the symbolic expression for the function. Nevertheless, we could see that the students in the group that used the calculator worked on their own, finding the zeroes of a polynomial and used the resource for changing the cuts interactively; so the students in the group that used calculators were more willing to produce a polynomial expression for *f*, even though this was not required for solving the problem.

Conclusions

We found the same phenomena that other researchers have observed.

First, that 'Technology, without curricula, is worth the silicon it is written on.' This was said by James Kaput at a conference of the National Council of Teachers of Mathematics, 1994. We need to define a curriculum, from the beginning, using the calculator as another resource for the class. Just putting the calculator as 'another peer,' even thinking of the need for controlling the variables, is useless for conducting an analysis of the students' change in their knowledge.

We need to allow the use of the resource on assessments. The restriction we used showed that the students used the calculator only for verifying procedures, instead of for formulating and testing conjectures. We have observed this last type of use of the calculator in groups that were allowed to use calculators all the time.

We need to find another way to conduct this type of exploration (see Dunham and Dick, 1994, for a deep analysis of the difficulties of this kind of studies). No one can control all the factors and variables involved in the classroom. We are convinced that a study case, although more specific and with fewer possibilities for making generalizations, would give us more information that this exploration gave in the two years we worked on it. Nevertheless, the things we have learned using this approach were indeed valuable: we began to change our view about mathematics and about its role in the society; we have challenged our view about teaching and learning processes; and we developed a new view about our students. That was the value of this exploration.

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[10] For the way the exercise was formulated, it was necessary to work procedurally on the graph of *f*.

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Appendix

Table 1 Table of perspectives. Meaning of the cells.

Perspective	Notation system: Graph in coordinate plane	Notation system: Symbolic
Operational	Plotting of points in the plane and then connect them. Identification of the coordinates of points that are on a graph.	Syntactical transformations only in the symbolic notation. Formulas without context. Symbolic expressions that are not used.
Structural	Movements of graphs: horizontal and vertical translations and dilatation without a symbolic or numeric reference.	Description of characteristics of objects using symbolic notation. Interpretation of the meaning of a parameter in a symbolic expression. Recognition of an expression as an identifier of a family of functions or as an element of the family of functions.

Table 2: Comparison of use of strategies.

		Group without calculators		Group with calculators	
Strategy	Cells visited	% use 1st test	% use 2ndtest	% use 1st test	% use 2ndtest
Give an expression (polynomial, rational or trigonometric) for *f*. *g* is written using '*f*(*x*)'; the translation and dilatation factors are recognized; there is a verbal description of what happened to *f*.	OG-OS-SS-SG		6		11
Give an expression (polynomial, rational or trigonometric) for *f*. *g* is written using this definition of *f*(*x*) and the translation and dilatation factors.	OG-SG-OS				11
Give an expression (polynomial, rational or trigonometric) for *f*. *g* is written using '*f*(*x*)'; the translation and dilatation factors are recognized.	OG-SS-OS		6		
Give an expression (polynomial, rational or trigonometric) for *f*.	OG-OS	6	11	11	22
Give an expression (polynomial, rational or trigonometric) for f. Give a verbal description of what happened to *f*; the translation and dilatation factors may have been recognized.	SG-OG				6
Give a table of values for f and use this for giving an expression for *f*.	OG		6		6
Give an expression for *g* using '*f*(*x*)'. The translation and dilatation factors are recognized.	OG-SS		22		22
Give a verbal description of what happened to *f*. The translation and dilatation factors are recognized.	SG-OG		16	6	6
Nothing.		94	27	83	16
Unable to classify.			6		

ON THE UTILIZATION OF ENCODING PROCEDURES ON THE TREATMENT OF GEOMETRICAL PROBLEMS[*]

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The analysis reported here is a part of a on-going study on the utilization of figure in geometry by junior high-school pupils. In this paper we analyse pupils' apprehension of the figure and the utilization of codes made by twenty 11-12 years-old pupils when they solve a geometrical problem. Codes appears as a powerful way to make appropriate links between the figure and the hypotheses of the problem and can, in this way, contribute to an effective geometrical reasoning.

This paper concerns the utilization of encoding procedures used by pupils when solving geometrical problems and its purpose is to contribute to the understanding of the rules underlying pupils'utilization of figures in solving geometrical problems ; it is a part of a more general on-going study on external representation in geometry. Before presenting our study, we will clarify some theoretical framework and assumptions about external representation[1] in geometry.

Theoretical framework

1. Status of a representation. External representation doesn't have always the same status in geometrical problems. In previous studies, we showed the importance of what we called the **status** of a representation (Mesquita, 1989, 1992, 1994, in press), which is related to the status of geometrical objects introduced by Husserl (1936/62). To Husserl, a geometrical object has one of this status : it can be a *finiteness*, in the sense of a finite and varied form in its spatio-temporality, or a geometrical form in its *ideal objectiveness*, detached from the material constraints linked to external representation.

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[*] We thank to Mrs. Line Eymery, from Ecole Paul Verlaine and IREM, Lille, for her participation and her interest in this study.

[1] In this paper, we use the term 'figure' as a synonym of external (materialized on a support, paper or other, by opposition to mental, or internal), and ikonical (or figurative, centered on visual image, by opposition to other possible semiotic systems) representation of a concept or a situation in geometry. By geometrical problem, we mean here a geometrical question posed to pupils, which is formed by a statement and a figure.

From a mathematical point of view, geometrical objects and their representation are generally considered as an ideal objectiveness, unless otherwise is expressed ; external representation is aprehended as a network of geometrical relationships between elements. From a mathematical point of view, the status of a represented object is less clear. Our studies, concerning pupils in junior high-school, suggest that a represented objet can have to pupils different status: a *finiteness*, or an *ideal objectiveness*, in the sense of Husserl, but also a status which appears as an intermediate between these ones, which we called *equivalence class ;* in this case, external representations conserve ratio and, in this sense, act as a support to proportionnality computations. The figure is treated by pupils as an element of an equivalence class (Mesquita, 1994; in press).
Let us consider the case of the rectangle represented in fig.1 :

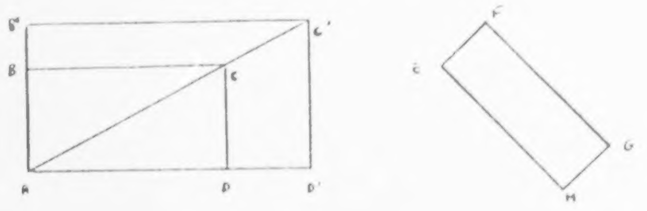

Figure 1

How many rectangles are considered in figure 1 ? One, in considering the rectangle as an ideal objectiveness (the representations ABCD, AB'C'D' and EFGH are three representations of the same ideal objectiveness), two - for the equivalence class status (ABCD and AB'C'D' being similar, are considered in this case as the same rectangle), or three finiteness rectangles.

2. Types of apprehension

According to Duval (1988, 1995), external representation can mobilize different types of apprehension, or form by which a solution of the problem can be suggested by the figure.
We will distinguish here the two forms of apprehension underlying a part of our analysis[2] : the **perceptive apprehension**, or the immediate and automatic apprehension of the figure, linked only to the figure, independently of the hypotheses mentionned in the statement. It is a form of apprehension associated with the

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[2] Other forms of apprehension are described in Duval (1988) and analysed in Mesquita (1989).

gestaltist laws of perception.

Another form of apprehension of a figure in geometry depends exclusively on the status accorded to propositions by the statement of the problem ; it is linked with the properties mentionned as hypotheses ; in this case, the figure appears as "a piece of a theoretical discourse" (Duval, 1988) : it is the **discoursive apprehension**. To apprehend discoursively a figure means to consider it as a network of properties given by hypotheses in the statement of the problem.

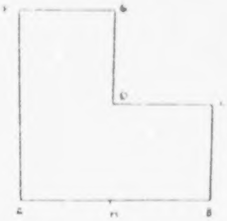

Figure 2

In the exemple given by fig. 2, the perceptive apprehension suggests the capital letter "L", or "a square from which another square had been removed",where M the half-point of AB : it is possible to describe it without any statement (J.C. Rauscher, cited in Mesquita, 1989).

A discoursive apprehension of the same problem is impossible to determine without the statement of the problem : M would be the half-point of AB, if the hypotheses mentionned in the statement clearly explicite it.

Definition of the problem and methodology

This study reports a part of a more general on-going study on the utilization of external representation by junior high-school pupils. External representations have a central importance on the learning of geometry, but they appear as a hidden part on this learning. Figures are used in different ways, including symbolic ones, but the rules of this symbolism remain implicit ; pupils use them in personal manners.

To study the personal utilization of figures, we prepare a program of intervention in a junior high-school class (twenty-five 7th graders, 11-12 years-old), from a school considered in a **difficult area** in Lille, France. Our main goal is to explicit the different ways how pupils apprehend and use the figure. This program of intervention is based on the assumption that encoding procedures is a mean of stimulating discoursive apprehension of the problem ; it includes the utilization of different kinds of representations (hand-made figures, ruler-made figures, sotfware-made figures, pictures), using differents supports (squared paper, computer screen, photographies). Our interventions, made in the framework of the class, is based on the work with different external representations ; we pass some questionnaries and made clinical interviews with pupils. We enregistrate some moments of our intervention, and analyse the protocols and other productions of pupils

We analyse here the first part of our intervention. In this phase, pupils are asked to solve some problems, in which encoding procedures can facilitate the resolution.
We analyse here pupils' answers to one of these problems :
These are the givens of a geometrical problem (fig. 3) :

- *1,2, and 3 are equilateral triangles,*
- *4 and 5 are rectangles,*
- *6,7 and 8 are squares.*
- *the length of AB is 4m.*

Can you find the lengths of QP ? of LJ ? of EF ? Explain your answer.

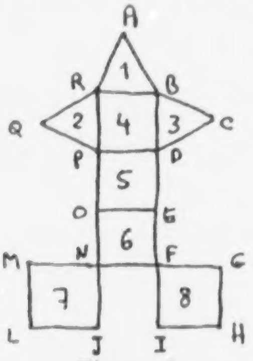

Figure 3

Pupils are asked to keep all the traces of their work, correcting but not erasing answers they judge incorrect ; their drafts were returned to us. We asked pupils to try to mark on the figure what it was known about the figure, with some physical marks or in any other way.

Analysis of task. The situation was specially designed to stimulate encoding procedures ; therefore, the task was based on the utilization of a) basic properties concerning triangles and quadrilaterals (rectangles and squares), b) a transitive property of equality. The statement and the figure are not congruent, in the sense that the information issued from the figure and from the statement are not the same (rectangles and squares are represented in a similar way), and in consequence, the problem needs of a discoursive apprehension to be correctly solved.

Results

Use of encoding procedures

In the following, we described the kind of encoding treatments used by pupils, according to the type of marks used and to the links between these codes made by pupils.

Types of encoding marks

Numerical marks. In this phase of the study, the great majority of pupils (twelve, from the twenty pupils) used almost exclusively numerical marks (measure or abstract number) : they write, near the correspondant sides, their measures, a number accompagnied by the respective unity, or only its value, sometimes with errors (see below). They use measure marks to express the measure of the length, i. e., to mention that the length equals 4m, writing "4m" or "4" near each side.

Abstract marks. In this study, graphic marks such as slashes, are used by two pupils. One pupil used (abstract) numbers as encoding marks : this pupil, Fatima, used the cipher "1" to designate the equality of lengths, near of the respective sides ; keeping a certain distance of the "1", she mentioned "4m". Another pupil, Samir, used abstract marks (slashes) and numbers to express, in a particular way, relationships between sides (see below Samir's utilization of slashes).

Errors observed in the encoding procedures

Pupils make errors of three types in coding the figure : perceptive errors, dues to making encoding marks based on a descriptive apprehension of the figure (for instance, coding a rectangle as it was a square). In fact, in these cases, the six pupils make errors in associating the given measure (4m) to sides from which the statement does not enable to conclude it.

Another type of errors concerns a bad report of encoding marks : they used **4cm** instead of **4m** ; in general, the two pupils doing this error, do it systematically.In this reporting errors, they write **"4cm"** (instead of **4m**) near the correspondant side (whose measure was different from).

A third group of errors concerns the code of their own figure : pupils code the figures according to the dimensions of their own figure, drawn by themselves ; for instance, they use numerical codes like "3cm". Only few pupils made this kind of errors - four pupils do it.

As a conclusion, we can say that the great majority of pupils of this study used numbers (with or without unit) as encoding marks. It seems that codes express, to these pupils, the length of the sides - rather than a relationship between segments or their length. The codage appears here as an indicator of lengths, as a particular property. In this sense, we will speak on **concrete codage**.

In this study, one pupil used numbers as a transition to higher elaborated encoding marks. In this case, encoding (symbolic) marks, express a relationship existing between sides or segments, or its lengths, which seems to be more abstract, more symbolic.

Links between marks

Connected codes. Almost all the pupils using encoding procedures made connections among their codes. codes in a linked way. In the given situation, the great majority of the pupils repeat systematically the length of the sides. In general, they repeat it correctly, based on the properties given by statement.

Independent, local marks. Dissociated codes. One of the pupils, Samir, make a different use of codes : he codes separately each subfigure, without making any linking among them. Different local codes are used by Samir : graphic marks (slashes in different quantity and different colours), in general without articulation between information (fig.4). This kind of code, without any treatment, asks for complements, i. e., the description of relationships between (common) segments : in this sense, we call it a **dissociated codes**.

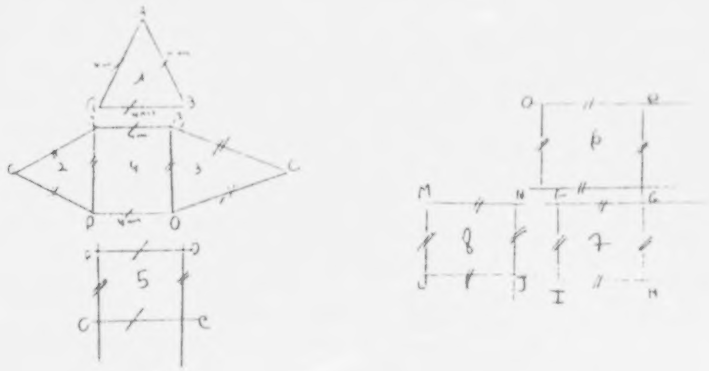

Figure 4

This pupil used measure codes, slashes and colours in the same figure. He considered encoding marks concerning each subfigure and coded it correctly, using numerical (measure) codes, slashes and different colours in the different figures ; however, he used the same manner (double coloured slashes) in coding independent squares, as if they were equal, which could not be determined by given conditions. However, he did not make the necessary association between subfigures and respective marks. He codes each subfigure, but he did not do the necessary association between the fact of some sides are common to several subfigures. He was able to answer correctly the answers, but not justifying it correctly.

Encoding procedures and apprehension

Perceptive apprehension. The great majority of the pupils - thirteen pupils- used encoding marks suggested by the perceptive properties of the figure; in this sense, they have a perceptive apprehension of the figure. In fact, most of them code the perceptive properties suggested by the figure of the problem, rather then the properties mentionned by the statement. It reveals that for these pupils perceptive apprehension of this figure is dominant. A small group had a apprehension of the same type, but they mixed in their justifications properties mentionned in the statement and properties issued from the figure.

Discoursive apprehension. Few pupils -three pupils- have, in this study, a discoursive apprehension of the figure.

Status of the figure

In this study,a great majority of the twenty pupils identify the external representation as a equivalence class. Their figure appears as to these pupils as a representant of this class. In this case, pupils use the figure as a support, or a basis, to proportionnality computation, and considered it invariant by similarity. It was the case of twelve pupils. Some others, a small group of six pupils use the figure as a finiteness, having difficulties in working with something different from their own representation of the geometrical problem. Only for two pupils, the external representation is considered as an ideal objectiveness.

Conclusions and discussion

Encoding procedures are used by 11-12 years-old in some personal manners : numerical codes (measure and abstract ones), graphic codes or mixed codes (using several types of marks, in a consistent way. In general these codes are used in an articuled, but personal way. The errors observed in the encoding procedures are in general issued from the perceptive apprehension of the figure. Some of these procedures are more abstract than others. The procedures more abstract have been observed in the pupils using mixed ways of encoding.

The mixte way of encoding, where numbers and graphic abstract marks are used simultaneously, suggests an emergence of symbolic conventions.

From a didactical point of view, this study gives some indications on the importance of doing in classes this kind of work with pupils, as a preliminary to a further introduction of conventions.

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CHILDREN'S DEVELOPING MULTIPLICATION AND DIVISION STRATEGIES

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60 female students were observed 4 times during Grades 2 and 3 as they solved the same set of 24 multiplication and division word problems. From the correct responses, various calculation strategies were identified and grouped into categories. It was found that the students used three main categories of calculation strategy: direct counting, repeated addition and multiplicative operation. A fourth category, repeated subtraction, only occurred in division problems. The results are interpreted as showing that children acquire an expanding repertoire of calculation strategies, and that the strategy they employ to solve any particular problem reflects the mathematical structure they impose on it.

Several recent studies have shown that students can solve a variety of multiplicative problems long before formal instruction on the operations of multiplication and division. For example, Kouba (1989) found that 30% of Grade 1 and 70% of Grade 2 students could solve simple equivalent group problems and Mulligan (1992) found a steady increase in success rate on similar problems from over 50% at the beginning of Grade 2 to nearly 95% at the end of Grade 3. More recently Carpenter, Ansell, Franke, Fennema and Weisbeck (1993) found that even kindergarten students could learn to solve multiplicative problems.

Of particular interest has been the solution strategies children use. Anghileri (1989), for example, classified the strategies used across multiplication problems with six different semantic structures as follows: physical modeling followed by unitary counting; rhythmic counting (e.g., 1, $\underline{2}$, 3, $\underline{4}$, 5, $\underline{6}$); skip counting (2, 4, 6); additive calculation (2 + 2 + 2); and multiplicative calculation (2 x 3). As she noted, the increase in sophistication from rhythmic counting to additive calculation seems to be more a result of children's deepening understanding of addition rather than any change in their conception of multiplication. The use of a multiplicative calculation, by contrast, implies a more advanced conception of multiplication as a binary operation.

Kouba (1989) investigated the solution strategies children use for both multiplication and division, but only in three semantic structures: equivalent groups for multiplication and partition and quotition for division. For our purposes, her most pertinent finding was that partition and quotition problems did not generate different calculation strategies. She reported, without citing specific data, that both types of division were solved using either repeated subtraction or repeated building up.

Children's division strategies have also been studied by Boero, Ferrari and Ferrero (1989), Murray, Olivier and Human (1992), and Bryant, Morgado and Nunes (1993), while Carpenter et al. (1993) comment in general terms on both multiplication and division strategies. However, no-one appears to have looked closely at the relation between multiplication and division strategies or to have investigated how children's strategies change over time.

The present study was designed to extend previous studies by including a wider variety of semantic structures for both multiplication and division. The following research questions were posed:

1. Can categories of strategy be identified which occur in children's solutions to both multiplication and division word problems?
2. Does the semantic structure of the problem influence children's strategies?
3. How do children's strategies change over time, especially as a result of instruction?

A previous paper (Mulligan 1992) described children's solution strategies in general terms. The present paper reports on a classification of their *calculation* strategies, considered independently of modeling strategies (such as concrete modeling, visualization and symbolic modeling). We decided to concentrate on calculation strategies because, although modeling strategies are important in practice, we feel that they probably reflect children's familiarity with a particular calculation strategy rather than any fundamental change in the way they structure their solution. Further details are to be found in Mulligan and Mitchelmore (in press).

Method

Six multiplication problems were constructed with five of the semantic structures identified by Vergnaud (1988), Greer (1992) and others: equivalent groups, multiplicative comparison, cartesian product, rectangular array and rate. Six division problems were constructed with quotition, partition, multiplicative comparison and rate structures. Each problem was written in two forms, one with the product between 4 and 20 and one with the product between 20 and 40. All problems were set in contexts which are familiar to young children, and all involved only whole number data and answers. The problems were written on cards and read aloud to children in a fixed order.

The sample consisted of 60 female students. Clinical interviews were conducted by the first author four times in two successive years when the students were in Grades 2 and 3. At the time of the first interview, students had received teacher instruction in basic addition and subtraction but not in multiplication and division. Between the third and fourth interviews, all students were given instruction in basic multiplication facts involving the 2- to 10-times tables but not in related division facts. During the interviews a number of small cubes were available on the table, but no paper or pencil; so students were forced to use either concrete modeling or mental calculation strategies, or both. Each interview was audiorecorded, and neutral prompts were used whenever the child's response was unclear.

Classification of calculation strategies

Since the vast majority of incorrect responses resulted from superficial strategies (mainly adding the two given numbers), it was decided only to consider correct responses. The first author initially coded children's calculation strategies. A research assistant then independently coded every fifth interview, with a 92% agreement rate.

The various calculation strategies were examined to identify underlying principles. It was found that they could be grouped into four categories corresponding very closely to those reported in previous research: direct counting (DC), repeated subtraction (RS), repeated addition (RA) and multiplicative operation (MO). DC, RA and MO occurred in both multiplication and division, but RS occurred only in division. All occurred with and without concrete modeling. Table 1 describes the various strategies in each category; we give some examples below.

Table 1: Classification of children's calculation strategies

Category	Calculation strategy
1. Direct counting (DC)	Unitary counting
	One-to-many correspondence*
	Sharing*
	Trial-and-error grouping*
2. Repeated subtraction (RS)	Rhythmic counting backwards*
	Skip counting backwards*
	Repeated subtracting*
	Additive halving*
3. Repeated addition (RA)	Double counting forwards
	Skip counting forwards
	Repeated adding
	Additive doubling
4. Multiplicative operation (MO)	Known multiplicative fact
	Derived multiplication fact

*Only found in division problems.

Direct counting (DC) strategies consist of modeling the problem (using either cubes or visualization) and counting the objects or groups of objects. DC strategies were frequently observed for both multiplication and division. For example, asked to find how many children sit at two tables, four to a table, Michelle put out three blue and five red cubes in two groups of four, said there were three boys and five girls, and calculated "3+5=8." Asked an inverse problem, Julia counted out the total number of cubes and dealt them out two at a time to the correct number of imagined tables. In DC the problem is essentially solved by the concrete materials themselves, the strategy not taking advantage of the equal sizes of the groups.

Repeated subtraction (RS) strategies—only observed for division problems—all start with the dividend and use a systematic calculation procedure in which the number in each group is repeatedly taken away. For example, to place 16 children 2 to a table, Amy counted out 16 cubes and then took away groups of 2 cubes, saying "16, 14, 12, 10, 8, 6, 4, 2, nothing left ... that's 8 tables." The distinction between Amy's strategy and a DC strategy is that Amy simultaneously counted both the number of cubes left and the number of groups already formed. All RS strategies create a sequence of multiples starting with the dividend.

Repeated addition (RA) strategies also take advantage of the equal-sized groups present in the problem situation and create a sequence of multiples, but starting with zero and ending with the product or dividend. Children seemed to use RA equally fluently for multiplication and division, including the partition division structure where the size of each group has to be guessed in advance. For division, RA may be more advanced than RS because it allows the same strategy to be used for both division and multiplication problems.

In *multiplicative operation* (MO) strategies, the student gives a correct response without appearing to form the entire sequence of multiples. Typical MO responses to multiplication problems were "I made one group of 3 and timesed it," and "It's three multiplied by four ... I multiplied it straight off." For division, the solution was often guessed and checked by multiplication; in other cases, the student appeared to search for a multiple of the divisor which was equal to the dividend. (Only a few students demonstrated an explicit awareness of division as an operation, mostly in a halving strategy.) We justify including the use of derived facts as an MO strategy because, even though addition is used, the basic aim is to calculate a product without creating the entire sequence of multiples.

Choice of calculation strategy

Figures 1 and 2 show how children's choice of calculation strategy in their correct responses to the multiplication and division problems changed from Interview 1 to Interview 4. Although children were only instructed in multiplication (starting between Interviews 2 and 3), the patterns for multiplication and division are surprisingly similar. In both cases, the most common correctly used strategy at each interview was repeated addition (15-30%); DC strategies remained more or less constant at about 10%; and MO strategies were rare in Interviews 1 & 2 (less than 3%) but increased dramatically by Interview 4 (to 28% for multiplication and 17% for division). The one strategy unique to division, RS, only led to correct responses in 4-12% of the solutions.

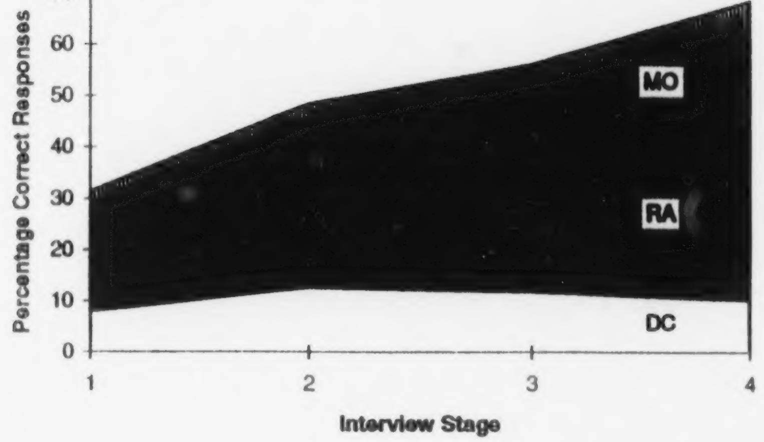

Figure 1. Percentage of sample giving correct responses to multiplication problems at each interview, classified by strategy.

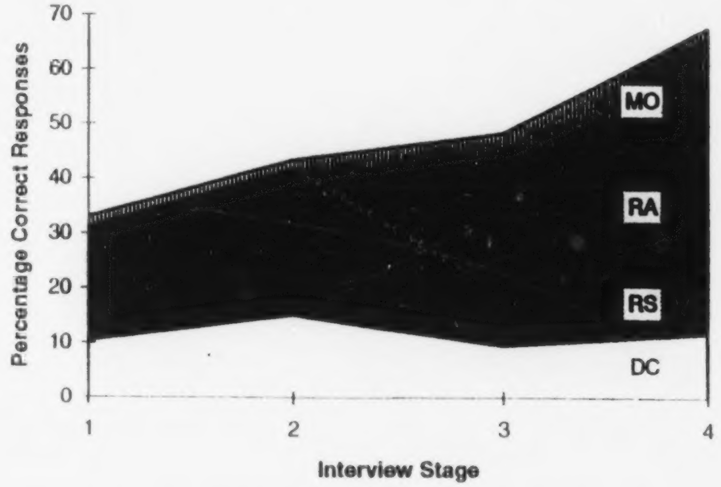

Figure 2. Percentage of sample giving correct responses to division problems at each interview, classified by strategy.

The size of the numbers in the problems had a fairly consistent effect on choice of strategy. For both multiplication and division, successful use of DC was more common for large numbers than small numbers (13% compared to 9%) whereas the reverse was true for both RA (18% compared to 32%) and MO (7% compared to 11%). Correct solutions of division problems using RS were also less frequent for the large number problems than the small number problems (5% compared to 11%). Many students who had successfully used RS or RA for a small number problem seemed to experience a processing overload when attempting to use the same strategy for the corresponding large number problem; they then often solved the problem using DC. Similarly, students who had used MO for a small number problem were often unable to retrieve the number fact required for the corresponding large number problem and reverted to RS or RA.

By contrast, the semantic structure of the problems had little consistent effect on choice of strategy. Only three problems consistently attracted DC strategies: array multiplication problem (22% of all responses), the equivalent groups multiplication problems with large numbers (20%), and the quotition division problem (23%). Correct use of RS strategies was only consistently common on the small number partition problem (31%); this was the only division problem easily solved by additive halving, which many of the students used.

As an example of inconsistency, consider the rate problems—most frequently solved successfully using RA for multiplication but MO for division. All four problems involved multiples of 5, which seem to be second only to multiples of 2 in terms of their familiarity to young children. In the small number division problem the multiplication fact 4 x 5 = 20 was frequently recalled, and on the large number division problem 8 x 5 was often derived from it by doubling. However, in the small number multiplication problem, doubling 5 was most often solved using a "5 and 5 makes 10" argument. In the large number multiplication problem, the odd multiple of 5 seemed to be relatively unfamiliar and students simply counted in 5's.

Another example: Despite the primacy of RA overall, MO was the most frequent correctly used strategy for the comparison multiplication problems. The cause seemed to be a linguistic cue in the problem statement, namely "Sue has 4 times as many"; for example, Lisa responded to this problem by saying "times as many ... that's multiply ... three fours."

Most students were not consistent in the strategies they used at any interview stage. At each interview, there were some students who used the same strategy on all problems but there were others who used as many as three different strategies. On the other hand, students showed a consistent progression in the strategies they used from interview to interview in the strategies they used for each problem. On all 12 multiplication problems, in only 3% of the cases did students successfully use a more primitive strategy to solve a problem which they had successfully solved at the previous interview and only 2% failed to solve it. For the division problems, the data confirm our earlier claim that RS is a more primitive strategy for division than RA: On all 12 division problems, there was not one single case where a successful use of RA was followed by successful use of RS at the next interview.

Discussion

Among students in Grades 2 and 3, we have been able to clearly identify three categories of calculation strategy used for multiplication (direct counting, repeated addition and multiplicative operations) and four for division (direct counting, repeated subtraction, repeated addition and multiplicative operations). As the names imply, there are basically only four strategies. Our data show a consistent progression in the strategies used by students in Grades 2 and 3, from direct counting to repeated addition or subtraction to multiplicative operations.

Previous findings that the calculation strategies children employ vary from problem to problem have been confirmed. However, the structure of the preferred strategy did not necessarily correspond to the semantic structure of the problems: All strategies were employed across all problems. Many of the observed differences in preferred strategy were readily explained by the size of the numbers, the particular multiples involved, and the language used to describe them.

We did not expect to find such a strong preference for a repeated addition strategy of division across all semantic structures. This phenomenon appears to be a result of the close connection which students see between division and multiplication problem situations before they receive instruction in division. The same close connection is evidenced by students' spontaneous use of an multiplicative operation strategies for division shortly after instruction in multiplication.

These conclusions are, of course, limited by the problems used in this study. In particular, the strong effects of multiples of 2 and 5 were not anticipated. A clearer picture of the variation in strategies would be obtained if the numbers were better controlled. Also, although we have included a wider range of semantic structures than previous studies, there are still many others which could be investigated.

Despite these limitations, our findings do not seem to be in agreement with Fischbein, Deri, Nello, and Merino(1985) who proposed that "each fundamental operation of arithmetic generally remains linked to an implicit, unconscious, and primitive intuitive model" (p. 4) and that "the structure of the problem determines which model is activated" (p. 7). Instead, it would seem that children develop a repertoire of increasingly efficient strategies (direct counting, repeated addition/subtraction, and multiplicative operations) which they can apply to both multiplication and division problems of all semantic structures. The strategy employed to solve a particular problem does not reflect any general problem characteristic but rather the mathematical structure which the student is able to impose on it.

Conclusions

The present study raises several questions about traditional approaches to teaching multiplication and division of whole numbers in elementary school. Children would surely benefit if teachers provided them with opportunities to solve multiplicative word problems as early as the first year of schooling, and if they linked multiplication and division much more closely. The teacher's task is to acknowledge that students use a wide variety of strategies and to encourage them to expand their repertoire. It would also seem possible to include multiplicative word problems involving rational numbers much earlier than at present. For example, Confrey and Smith (1995) describe a broad category of measurement situations which appear familiar to young children and easily extend into rational numbers. Also, Behr, Harel, Post and Lesh (1994) show how rational number arithmetic can be approached in such a way as to make the connection with whole numbers explicit. Only further research will reveal whether such approaches are likely to be more successful than traditional methods.

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CHILDREN'S CONCEPTS OF TURNING: DYNAMIC OR STATIC?

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To investigate how children conceptualise and classify various turning situations, 36 children selected from Grades 2, 4 and 6 were questioned on the similarities they saw between an oven temperature knob, a door, and bends in a road. Between Grades 2 and 6, there was a general increase in children's tendency to recognise angle-related similarities. The data suggest that children conceive of turning around a central point as essentially the same movement as turning around a hinge. Turning around a bend is conceived differently: The similarity to turning around a point or hinge is usually based on the static appearance of the angles and not on the dynamic way they are formed. Implications for the early teaching of angle are explored.

For some time now, mathematics educators have recommended treating angle in terms of turning, at least in the elementary stages (Wilson & Adams, 1992). This view has found general acceptance; for example, our local mathematics syllabus for Grades K-6 refers to angle as "the amount of turning between two lines about a common point" (New South Wales Department of Education, 1989, p. 79). Defining angle in terms of turning is intended to stress the relation between the arms of the angle and to facilitate angle measurement.

However, two studies we carried out recently (Mitchelmore, in press; Mitchelmore & White, 1995) suggest that the above definition might not be appropriate for young children. In these studies, children in Grades 2 and 4 were presented with models of a number of physical angle situations (a doll turning about a fixed axis, a variable hill, a pair of scissors, a map of some bending roads, a ball game, and floor tiles). They were then questioned on their understanding of each situation and the relation between them. The children were surprisingly good at visualising rotations of the doll, but several findings suggested that they did not interpret turning as an angle:

- Many children represented the turning doll by a single turning line. Only 54% of the responses in Grade 2 and 76% in Grade 4 represented the rotation with an angle.
- Few children could represent turns in a drawing. Only 8% of the Grade 2 children and 33% of the Grade 4 children showed a turn by two radii joined by an arc or arrow.
- Few children recognised appropriate angle-related similarities between the turning doll and the other physical angle situations presented.
- Although the Grade 4 children had all studied angle as amounts of turning in school, *none* stated this definition of angle or gave turns as examples of angles. Furthermore, 42% explicitly stated that the turning doll situation did not involve angles.

Further evidence that turning is a more difficult and complex concept than has often been assumed is provided by the common finding that children learning LOGO cannot easily relate the turn parameter to the angle formed by the turtle's path (Clements & Battista, 1992). Clearly children's concepts of turning deserve closer investigation.

A tentative classification of turning situations

Results from our previous studies suggest that young children might classify turning situations as follows:

- *Unlimited rotations* about an interior point. Such objects (revolving doors, fans, merry-go-rounds, wheels, and so on) often possess rotational symmetry.
- *Limited rotations* about an interior point. In such situations (e.g. control knobs, door handles), there are well-defined limits between which the turning occurs.
- *I- hinges* (such as normal doors and most instrument pointers), where a single, linear object is hinged about one end and can rotate between well-defined limits.
- *V-hinges* (such as pocket knives and book covers), where two linear objects are hinged about a common end-point.
- *X-hinges* (such as scissors and latticework), where two linear objects are hinged about a common interior point.
- *Bends* which can be regarded as two line-segments with a common end-point (as distinct from bends which are better modelled by a continuous curve). In contrast to V-hinges, the turning movement in this situation occurs at the common end-point during a forward movement along the path formed by the two line-segments.

This classification clearly needs validation from further empirical data.

The present study

The present study was designed to investigate how well children recognise the similarities between different turning situations. Three situations which previous studies suggested might be conceptualised differently were selected for closer study: limited rotations, I-hinges and bends. The basic research question was:

- What angle-related similarities do children recognise between the three turning situations?

Children of different ages were included in order to investigate how the recognition of similarity changes over time. The study was part of a larger investigation and only those aspects related to similarity recognition are reported here.

METHOD

Subjects

36 children from two Catholic schools in the north of Sydney participated in the study. There were 6 children (3 boys and 3 girls) from each of Grades 2, 4, and 6 at each school. Median ages were 7.1 yr, 9.5 yr and 11.3 yr respectively.

Materials

Realistic models were constructed of an oven temperature knob (limited rotation), a door (I-hinge), and a road network (bends). The knob could rotate clockwise through about 300° and the door could open through 180°. The road map showed two routes between a "factory" and a "hospital", each consisting of four straight segments rounded off slightly at their joins; the children were asked to drive a toy ambulance as fast as they could along both routes.

Procedure

Interviews were conducted individually and were audio-taped to assist in the data analysis. After a lengthy introduction designed to allow children to become thoroughly familiar with the movements in the three models, children were asked if there was "anything the same" about each pair of models. (The pairs were presented in the order knob-door, knob-roads, door-roads). Neutral prompts such as "Is there anything else?" were used until children could think of no further similarities.

When children gave global responses to the first pair, such as "The knob and the door both turn," the interviewer closed the door, turned the knob through about 45° from the "off" position, and asked the child to turn the door the same amount. When children gave global responses to the second or third pair, such as "The knob [door] and the road both turn," they were asked to turn the knob or open the door by the same amount of turning as is required to drive around the depicted 45° road bend (i.e., a turn of 135°). When children gave analytical responses to any pair, such as "They both have two lines meeting at a point," they were asked to indicate the two lines and the point on both physical models.

RESULTS

Tables 1 to 3 show the percentages of the sample who recognised some similarity between each pair of situations, the reasons they gave, and the accuracy with which they matched turns in the two situations. Many of the children gave several reasons why the two situations were similar, including reasons (both made of wood, painted white, and so on) which are irrelevant to the angle concept. All percentages are based on a maximum of 12 responses in each cell; because of multiple responses, the percentages giving each reason do not necessarily add up to the percentage recognising similarity.

The frequency with which the knob and the door were judged similar (Table 1) increased steadily from Grade 2 to Grade 6. Most of the children recognised a similarity between the two movements, and almost all of them could match a door turn to a knob turn with a reasonable degree of accuracy. However, relatively few children spontaneously referred to the movement as turning. A very few children stated that both situations involved two lines meeting at a point; all these children also recognised a similarity between the movements of the knob and the door.

Table 1

Percentage distribution of similarities recognised between knob and door

Item	Gd 2	Gd 4	Gd 6
Knob and door are "[sort of] the same"	58	75	92
Reasons given:			
Both turn	25	25	25
Both open and close	8	17	25
Both move	25	8	33
Both show two lines meeting at a point	0	8	17
Irrelevant features only	0	25	8
45° knob turn shown by door opening 45° ± 30°	58	67	83

Table 2 and 3 show that children recognised a similarity between the bends and the knob or the door about as often as they recognised the similarity between the knob and the door. However, there were in these cases far fewer explanations in terms of movement and more in terms of lines meeting at a point. Also, there was a larger proportion of older children who gave only irrelevant explanations.

Table 2

Percentage distribution of similarities recognised between bends and knob

Item	Gd 2	Gd 4	Gd 6
Bends and knob are "[sort of] the same"	58	67	83
Reasons given:			
Both turn	42	33	25
Both make angles	0	25	33
Both show two lines meeting at a point	0	8	33
Irrelevant features only	17	8	25
45° bend shown by knob turning 135° ± 30°	17	8	25

Table 3

Percentage distribution of similarities recognised between bends and door

Item	Gd 2	Gd 4	Gd 6
Bends and door are "[sort of] the same"	50	75	92
Reasons given:			
Both turn	17	17	17
Both open	0	25	0
Bend is like an open door	17	25	25
Both make angles	0	8	33
Both show two lines meeting at a point	0	17	33
Irrelevant features only/no reason given	17	25	25
45° bend shown by door opening 135° ± 30°	8	25	25

Children had great difficulty showing the sizes of the turn needed to drive around the road bend by turning the knob or the door. Of the children who claimed to see a similarity between bends and the knob, only 24% correctly made a 135° turn of the knob; 44% showed a 45° turn, 20% showed 315° and the remainder (12%) showed 225°. Of the children who claimed to see a similarity between bends and the door, only 27% made a 135° opening of the door; 65% made a 45° opening and the remainder (8%) did not know what to do.

DISCUSSION

Most of the young children in the present sample claimed to recognise a similarity between knobs, doors, and bends, the percentage increasing from about 50% in Grade 2 to about 90% in Grade 6. However, the percentage of the similarity explanations which were angle-related was somewhat greater for the knob-door than for the bends-knob and bends-door comparisons; and size matching was also far more accurate for the knob-door comparison.

A further difference lies in the type of angle-related similarities recognised. Two viewpoints can be distinguished: In some cases, a pair of situations was perceived as similar because of the turning movement involved—a *dynamic* similarity. In other cases, the perceived similarity was based on a configuration of two lines meeting at a point—a *static* similarity. For example, in Table 3, "both turn" and "both open" are clearly dynamic similarities while "both show two lines meeting at a point" is clearly static. From the interview tapes, it seemed that "the bend is like an open door" and "both make angles" were intended to express static similarities.

Table 4 gives the percentage of the sample giving each type of response at each grade level. (Because of multiple responses, the percentages giving dynamic and static responses do not always add up to the percentage giving an angle-related response.) The knob-door comparison evoked a large percentage of dynamic similarities at all ages, but the bends-knob and bends-door comparisons evoked a smaller and steadily decreasing percentage of dynamic explanations and a corresponding increase of static explanations. By Grade 6, all children who recognised an angle-related similarity gave (at least) a dynamic explanation for the knob-door comparison and a static explanation for the bends-knob and bends-door comparison.

We note also from the data on accuracy of size matching that, although most children seemed to compare the knob and door on the basis of the dynamic angles formed in both situations, they compared the dynamic angles formed by the knob and door with the static angle formed by the bend.

These results strongly suggest that young children conceptualise knobs and doors (and presumably other examples of rotation and hinging) quite differently from bends, the differences becoming clearer as children grow older. Children in Grade 6 have come to regard knobs and doors as essentially the same in that they both involve a similar, easily-related movement. However, they regard bends differently: They are not so sure whether bends are similar to knobs and doors, and if they are, it is usually because they show a similar configuration and not because they involve a similar movement.

Table 4

Percentage distribution of angle-related similarities recognised in three comparisons

Measure	Gd 2	Gd 4	Gd 6
Knob-door comparison			
Percentage recognising angle-related similarity	58	50	83
Reasons given:			
dynamic	58	50	83
static	0	8	17
Bends-knob comparison			
Percentage recognising angle-related similarity	42	58	58
Reasons given:			
dynamic	42	33	25
static	0	33	58
Bends-door comparison			
Percentage recognising angle-related similarity	33	50	67
Reasons given:			
dynamic	17	42	17
static	17	50	67

It is clear that each situation (rotations, hinges and bends) can be interpreted dynamically *and* statically. It would appear that young children spontaneously focus on the dynamic aspect of rotations and hinges, but on the static aspect of bends. To put it another way, they regard rotations and hinges as predominantly dynamic and bends as predominantly static.

We may speculate even further: The data suggest that children up to Grade 6 will have difficulty recognising similarities between every pair of physical angle situations where one is predominantly dynamic and the other is predominantly static. Furthermore, children are more likely to perceive such similarities by interpreting the predominantly dynamic situation statically (by identifying the two lines that constitute the angle, namely the starting and finishing position) rather than interpreting the static situation dynamically (by visualising or remembering a turning movement between the two given lines).

CONCLUSIONS

The above findings further question the received wisdom of teaching angle as an amount of turning. Certainly many significant everyday angle situations do involve turning, and children do come to recognise this (without necessarily calling the movement turning, however). But there are many other significant angle situations which children do not naturally interpret dynamically. Furthermore, many young children do not spontaneously conceptualise turning as a relation between two lines through a point, as is required in the definition of angle as "the amount of turning between two lines about a common point." Forcing children to interpret all angle situations in terms of turning is therefore unlikely to be an efficient teaching approach.

A more successful strategy is likely to be one which follows children's natural development but seeks to strengthen it by making the implicit concepts more explicit and by making stronger connections between different angle experiences. In particular, children need to learn how to interpret predominantly dynamic situations statically as well as interpreting predominantly static situations dynamically.

It would seem that teaching young children to interpret a bend in terms of movement along a path, and then focusing on the turn at the vertex, is an inefficient means of relating dynamic and static angle interpretations. All the evidence from this study and the literature cited earlier points to the difficulty of identifying the turn at the vertex and the even greater difficulty of relating its size to the size of the static angle. Using LOGO to teach about angles might better be left until after children have firmly established a link between static and dynamic interpretations of more familiar angle situations, including cases where the two interpretations lead to different angles.

The conclusions of the present study are limited by the small sample size, the restricted age range and the small number of contexts investigated. A larger study supported by the Australian Research Council is currently underway to test whether our interpretation of the present data is valid and to extend the conclusions to further angle contexts.

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